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MULTIPLICITY ONE THEOREMS

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1. Let $G = \mathbf{GL}_n$ over a global field k. We shall discuss the following two results.

(1) MULTIPLICITY ONE THEOREM. *Let π be an irreducible smooth admissible representation of $G(\mathbf{A})$. Then the multiplicity of π in the space of cusp forms is equal to one or zero.*

(2) Recall that any irreducible admissible smooth representation π can be written $\pi = \bigotimes_p \pi_p$, where each π_p is an irreducible admissible smooth representation of the local group G_p.

STRONG MULTIPLICITY ONE THEOREM. *Let $\pi_1 = \bigotimes_p \pi_{1,p}$ and $\pi_2 = \bigotimes_p \pi_{2,p}$ be two irreducible representations; suppose $\pi_{1,p} \cong \pi_{2,p}$ for every $p \notin S$, where S is a finite set, which in case $n > 2$ is assumed to contain only finite places. Then $\pi_{1,p} \cong \pi_{2,p}$ for all p. (Hence $\pi_1 = \pi_2$.)*

We begin by sketching the proof of the first Theorem (1). The basic tool is the Whittaker model. We introduce this first in the case k a local field, and (π, V) an irreducible smooth representation. In the case of k archimedean, we mean by "smooth representation" the representation of G on the space V of C^∞-vectors in some Hilbert space H on which G acts unitarily; for k nonarchimedean, this notion was introduced in Cartier's lectures. Let ψ be an additive character of k. Let

$$X = \begin{pmatrix} 1 & & * \\ & 1 & \\ 0 & & 1 \end{pmatrix}$$

be the standard maximal unipotent subgroup of G. Then a Whittaker model $W(\pi, \psi)$ for (π, V) is the image of V under an element of $\operatorname{Hom}_G(V, \operatorname{Ind}_X^G(\psi))$ where $\psi(x) = \psi(x_1 + \cdots + x_{n-1})$ if

$$x = \begin{pmatrix} 1 & x_1 & & * \\ & & x_2 & \\ & & & x_{n-1} \\ 0 & & & 1 \end{pmatrix}.$$

More explicitly, it is given by a set of smooth functions $\{W_v: G \to \mathbf{C}, v \in V\}$ for which

(i) $W_v(xg) = \psi(x) W_v(g)$, for all $x \in X$, $g \in G$.

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(ii) $W_{\pi(h)v}(g) = W_v(gh)$, for all $g, h \in G$.

We have the following important result due to Gelfand-Kazhdan for the case k nonarchimedean, and Shalika for general local fields ([1], [2]).

UNIQUENESS THEOREM. *For each irreducible admissible smooth representation* (π, V), *there exists at most one* $W(\pi, \psi)$ *(for fixed* ψ*).*

For k archimedean we assume that (π, V) is a unitarizable representation and $V = \{x \in H \mid (\mathscr{D}x, \mathscr{D}x) < \infty \ \forall \ \mathscr{D} \in \text{enveloping algebra}\}$. Here H means the completion of V with respect to the inner product (x, x). We assume also that $W_v(1)$ is a continuous linear functional on V with respect to the topology defined by seminorms $(\mathscr{D}x, \mathscr{D}x)$, $\mathscr{D} \in$ enveloping algebra.

Returning to the global case, we point out that the preceding discussion easily implies uniqueness of global Whittaker models (defined in the obvious way).

2. Global Fourier analysis. Let (π, V) be admissible irreducible cuspidal as before, $\varphi \in V$. Then we can define

$$W_\varphi(g) = \int_{X_k \backslash X_A} \varphi(xg)\psi^{-1}(x)\, dx.$$

Global Fourier analysis says that this "Fourier transform" defines a cusp-form uniquely. In the classical setting this is due to Hecke; for $n = 2$ it is proved in Jacquet-Langlands [3]; for $n > 2$ it is due independently to Piatetski-Shapiro [4] and Shalika [2]. The proof is motivated by a corresponding result over a finite field due to S. I. Gelfand [5]

It is now easy to see that these results imply Theorem (1), since

$$\dim \operatorname{Hom}_G(V, W(\pi, \psi)) = 1 \geqq \dim \operatorname{Hom}_G(V, L_0^2).$$

We now turn to the proof of the strong multiplicity one theorem. First we discuss the case $n = 2$; we need the following

SMALL LEMMA. (1) *Assume k local, (π_1, V_1), (π_2, V_2) two irreducible admissible representations with Whittaker models. Then there exist $v_1 \in V_1$, $v_2 \in V_2$ such that*

$$W_{v_1}\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} = W_{v_2}\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \quad (W_{v_i} \in W(\pi_i, \lambda)).$$

(2) *If $k = \boldsymbol{R}$ or $\boldsymbol{C}$ we assume that (π_1, H_1) and (π_2, H_2) are irreducible infinite-dimensional unitary representations. Denote by V_1 (V_2) the set of all smooth vectors in H_1 (H_2). Then there exist $v_1 \in V_1$, $v_2 \in V_2$ such that*

$$W_{v_1}\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} = W_{v_2}\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}, \qquad W_{v_i} \in W(\pi_i, \psi).$$

PROOF. For k a local nonarchimedean field it is known that V contains all Schwartz-Bruhat functions with compact support in k^*. Hence we have what we want.

Now let $k = \boldsymbol{R}$ or $\boldsymbol{C}$. The Kirillov theorem (see [8, p. 221]) says that each irreducible infinite-dimensional unitary representation of GL(2, k) remains irreducible after restriction on the subgroup $\{(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix})\} = P$ and hence as a representation of P is isomorphic to the standard representation of P. Hence, if $\varphi(x)$ is a C^∞-function with compact support then there exist $v_1 \in V_1$, $v_2 \in V_2$ such that

$$W_{v_i}\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} = \varphi(x).$$

REMARK. Assume that for a unitary representation with a Whittaker model the inner product can be written as an integral similar to the case for $n = 2$. Using this result we can prove the "small lemma" for any n as we did for $n = 2$. This implies the strong multiplicity one theorem for any n.

Next we give the formula for recovering φ from its Whittaker model due to Jacquet-Langlands, for $\mathbf{GL}(2, A)$:

$$(*) \qquad \varphi(g) = \sum_{\lambda \in k^*} W_\varphi\left(\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} g\right).$$

Now suppose π_1, π_2 satisfy the hypotheses of the theorem. To prove the assertion, it is enough to produce a nonzero $\varphi \in V_1 \cap V_2$, since then the irreducibility of (π_i, V_i) implies equality. Further, since $B_k \backslash B_A$ is dense in $G_k \backslash G_A$, it is enough to produce two functions (nonzero) $\varphi_i \in V_i$ which are equal on B_A (as usual B is the group of upper triangular matrices).

From the properties of Whittaker models and (*), it is enough to produce Whittaker functions W_1, W_2 such that $W_1((\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix})) = W_2((\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}))$, $x \in A^*$. One can suppose such W_i are of the form $\prod_p W_i^p$ and then it suffices to construct the appropriate W_i at a finite number of places (by assumption). But then one can use the small lemma. This type of argument was found independently here by Shalika and in Moscow.

For $n \geqq 3$, we need a similar small lemma (Gelfand-Kazhdan): Suppose k local, nonarchimedean, (π_i, V_i), $i = 1, 2$, irreducible admissible representations with Whittaker models. There exist $v_i \in V_i$ such that

$$W_{v_1}\begin{pmatrix} h & 0 \\ 0 & I \end{pmatrix} = W_{v_2}\begin{pmatrix} h & 0 \\ 0 & I \end{pmatrix}, \quad \text{all } h \in \mathbf{GL}(n-1).$$

One can then employ induction using arguments similar to the case $n = 2$, in order to prove the general case. It should be possible to prove this lemma also for k archimedean; then the restriction we made that S contains no infinite places could be removed.

Now suppose G is quasi-split and satisfies the *transitivity condition*:

$T(A)$ acts transitively on $\prod_{\alpha \text{ a simple root}} X_\alpha^*(A)$. Here T is a maximal k-torus in a Borel group, $X_\alpha^* = X_\alpha - \{I\}$ where X_α is the root group associated to the simple root α.

Define an automorphic cuspidal irreducible representation (π, V) to be *hypercuspidal* (degenerate cuspidal) if

$$W_\varphi(g) = \int_{X_k \backslash X_A} \varphi(xg)\psi^{-1}(x)\, dx = 0$$

for all $\varphi \in V$. Holomorphic cusp forms lifted from symmetric spaces which contain no copies of $H = \{\operatorname{Im} z > 0\}$ are of this type.

A cuspidal automorphic form will be called *generic* if it is orthogonal to all hypercuspidal automorphic forms (under the usual scalar product $\int_{CG_k \backslash G_A} \varphi\psi\, dg$).

Counterexamples to the Ramanujan conjecture given during this conference by Howe and the author are hypercuspidal forms [6]. The author does not wish to kill all belief in the Ramanujan conjecture; he conjectures it to be true for the generic cuspidal automorphic irreducible representation.

Now I shall sketch the proof of the multiplicity one theorem for generic cuspidal automorphic forms. First, the uniqueness theorem for local Whittaker models is true **[2]**. But of course now the Whittaker function does not define an arbitrary cusp form uniquely. Now it follows immediately from the definition that a generic cusp form is uniquely defined by its Whittaker function. This implies multiplicity one just as before. It can be proved that for each quasi-split reductive group there exist generic cusp forms. It also can be proved that for such groups there exists a unipotent subgroup U such that $\int_{U_k \backslash U_A} \varphi(ug)\, du$ can be expressed in terms of the Whittaker function. Since for any group except $\mathrm{GL}(n)$ there exist hypercuspidal forms, we cannot of course expect to be able to recover φ itself from its Whittaker function. Details concerning this will be given in a forthcoming publication of Novodvorskii and the author. (Notes prepared by B. Seifert and L. Morris.)

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