



Eisenstein series

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Let

$$\mathcal{H} = \{x + iy : y > 0\}$$

be the upper half-plane. It is a symmetric space. The group $G = SL_2(\mathbb{R})$ acts by Möbius transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d}$$

The measure $\frac{dx \, dy}{y^2}$ is invariant under G . The Laplacian is the second order differential operator given by

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

It is positive-definite and commutes with the action of G ; any other differential operator which commutes with the G -action is a polynomial in Δ .

Spectral analysis on \mathcal{H} .

Let $W_s(z) = \sqrt{y}K_s(2\pi y)e^{2\pi i x}$, $s \in \mathbb{C}$. This is an eigenfunction for Δ with eigenvalue $\frac{1}{4} - s^2$. Similarly, $W_s(rz)$, $r > 0$.

Any $f \in \mathbb{C}_c^\infty(\mathcal{H})$ can be expanded as

$$f(z) = \frac{1}{2\pi} \int_0^\infty \int_0^\infty (f, W_{it}(r\cdot))_{\mathcal{H}} W_{it}(rz) t \sinh \pi t dt \frac{dr}{r}$$

(corresponding to $G = NAK$) or, as

$$f(z) = \sum_{m \in \mathbb{Z}} \int_0^\infty (f, U_{it}^m) U_{it}^m(z) t \tanh \pi t dt$$

U_s^m - given in terms of Legendre function (corresponding to $G = KAK$; especially useful for K -invariant f 's, i.e. those depending on $\rho(z, i)$, where ρ is the hyperbolic distance).

The subgroup $\Gamma = SL_2(\mathbb{Z})$ is discrete in G with $\text{vol}(\Gamma \backslash G) < \infty$, i.e. it is a *lattice*.

Eisenstein series

$$\begin{aligned} E(z; s) &= \sum_{(m,n)=1} \frac{y^{s+\frac{1}{2}}}{|mz+n|^{2s+1}} \\ &= \sum_{\Gamma_\infty \backslash \Gamma} y(\gamma z)^{s+\frac{1}{2}} \quad z \in \mathcal{H} \end{aligned}$$

where $\Gamma_\infty = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}$.

Sometimes it is also convenient to use the normalized Eisenstein series

$$\begin{aligned} E^*(z; s) &= \zeta^*(2s+1)E(z; s) \\ &= \pi^{-(s+\frac{1}{2})} \Gamma\left(s+\frac{1}{2}\right) \sum_{(m,n) \in \mathbb{Z}^2 \setminus (0,0)} \frac{y^{s+\frac{1}{2}}}{|mz+n|^{2s+1}} \end{aligned}$$

(by pulling out $\gcd(m, n)$) where

$$\zeta^*(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s) = \zeta^*(1-s)$$

Properties

1. converges for $\operatorname{Re} s > \frac{1}{2}$.
2. $E(\gamma z; s) = E(z; s)$ for all $\gamma \in \Gamma$.
3. $\Delta E(\cdot; s) = (\frac{1}{4} - s^2)E(\cdot; s)$
4. Analytic continuation to $s \in \mathbb{C}$ (except for a simple pole at $s = \pm \frac{1}{2}$) and a functional equation

$$E^*(z; -s) = E^*(z; s)$$

5. The residue at $s = \frac{1}{2}$ is identically 1.
6. The Fourier expansion at the cusp

$$\sum_{r \in \mathbb{Z}} a_r(y, s) (= \int_0^1 E^*(x+iy; s) e^{-2\pi i r x} dx) e^{2\pi i r x}$$

is given by

$$a_r(y, s) = 4 |r|^s \sigma_{-2s}(|r|) \sqrt{y} K_s(2\pi |r| y) \quad r \neq 0$$

$$a_0(y, s) = 2\zeta^*(2s + 1) y^{s + \frac{1}{2}} + 2\zeta^*(1 - 2s) y^{-s + \frac{1}{2}}$$

where

$$\sigma_t(n) = \sum_{d|n} d^t$$

is the divisor function and

$$K_s(y) = \frac{1}{2} \int_0^\infty e^{-y(t+t^{-1})/2} t^s \frac{dt}{t} = K_{-s}(y)$$

is the K-Bessel function

We can write the Eisenstein series as Epstein zeta function

$$E^*(z; s - \frac{1}{2}) = \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{0,0\}} Q_z(m, n)^s$$

w.r.t. the binary quadratic form $Q_z(x_1, x_2) = |zx_1 + x_2|^2$

The holomorphy of $E(z; s)$ for $\operatorname{Re}(s) = 0$ and the formula for the first Fourier coefficient already imply that $\zeta(1 + it) \neq 0$ for all $t \in \mathbb{R}$, i.e. the Prime Number Theorem!

Special values:

$$E^*(i; s) = 2^s \zeta_{\mathbb{Q}(\sqrt{-1})}^*(s + \frac{1}{2})$$

More generally, let $z \in \mathcal{H}$ be a CM point of discriminant $d < 0$, i.e.

$$az^2 + bz + c = 0, \quad a, b, c \in \mathbb{Z}, b^2 - 4ac = d$$

Assume that d is fundamental, that is d is square-free except for 4. Then Γ_z corresponds to the ideal class \mathfrak{a} of $(a, (b + \sqrt{d})/2)$ in the ring of integers of $\mathbb{Q}(\sqrt{d})$ and

$$E^*(z; s) = \left(\frac{2\pi}{\sqrt{|d|}}\right)^{-(s+\frac{1}{2})} \Gamma(s + \frac{1}{2}) \zeta_{\mathfrak{a}}(s + \frac{1}{2})$$

Thus,

$$\sum_{z \in \Lambda_d} E^*(z; s) \chi(z) = \frac{1}{2} \sqrt{|d|}^{s+\frac{1}{2}} L^*(\chi, s + \frac{1}{2})$$

where Λ_d is the set of Γ -orbits of CM points of discriminant d and χ is an ideal class character.

Bernstein's proof of the analytic continuation of Eisenstein series

Lemma 1. For $\operatorname{Re}(s) > \frac{1}{2}$ $E(z; s)$ is the unique automorphic form F satisfying

1. $\Delta(F) = (\frac{1}{4} - s^2)F$

2. $F_U = y^{s+\frac{1}{2}} + *y^{-s+\frac{1}{2}}$ for some constant $*$ where $F_U(y) = \int_0^1 F(x + iy) dx$. Alternatively,

$$y \frac{d}{dy} (F_U - y^{s+\frac{1}{2}}) = (-s + \frac{1}{2})(F_U - y^{s+\frac{1}{2}}).$$

Proof. Consider $f = F - E(z; s)$. Then $f_U = *y^{-s+\frac{1}{2}}$ and therefore f is square-integrable. Since Δ is positive-definite, this implies that $\frac{1}{4} - s^2 \geq 0$, which contradicts the assumption that $\operatorname{Re}(s) > \frac{1}{2}$. \square

General principle: Suppose that S is a connected complex manifold and V a topological vector space. Let $\Xi = \Xi(s)_{s \in S}$ be a family of systems of linear equations in V depending holomorphically on S . That is, there exist analytic functions $c_i : S \rightarrow \mathbb{C}$ and $\mu_i : S \rightarrow V'$, $i \in I$ such that the system $\Xi(s)$ has the form

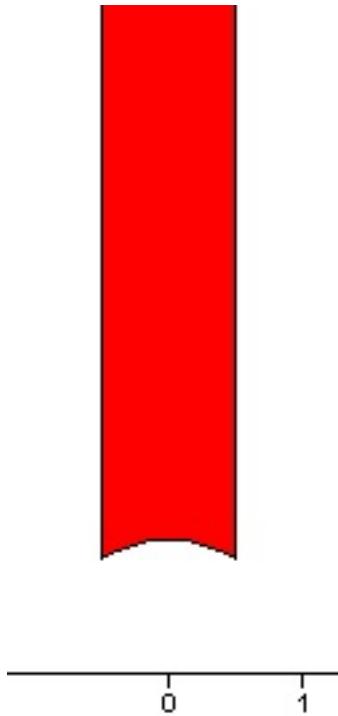
$$(\mu_i(s), v) = c_i(s).$$

Denote by $\text{Sol}(\Xi(s))$ the set of solutions of the system $\Xi(s)$ in V . Suppose that for some open $U \subset S$ (in the complex topology) the system $\Xi(s)$ has a unique solution $v(s) \in V$. Suppose further that Ξ is of locally finite type, i.e., for every $s \in S$ there exists a neighborhood W , a finite-dimensional vector space L and an analytic family of linear maps $\lambda(s) : L \rightarrow V$ such that $\text{Sol}(\Xi(s)) \subset \text{Im } \lambda(s)$ for all $s \in W$. Then $\Xi(s)$ has a unique solution $v(s)$ on a dense open subset of S and $v(s)$ extends to a meromorphic function on S .

Proof. Let S_0 be the set of points $s \in S$ for which there exists a neighborhood on which $\Xi(s)$ has a unique solution. We will show that $\overline{S_0}$ is open and that $v(s)$ is meromorphic on $\overline{S_0}$. By connectedness, this will imply the statement. Now, let $s \in \overline{S_0}$ and W, L, λ as above. We show that W (or alternatively, a dense open subset of W) is contained in $\overline{S_0}$. Upon passing to a subspace of L , we may assume that $\lambda(s)$ is monomorphic for all $s \in W$. The system $\Xi(s)$ induces a system $\Xi'(s)$ on L which has a unique solution $v'(s)$ on the non-zero open subset $W \cap S_0$. Then some $k \times k$ -determinant $D(s)$ of coefficients of $\Xi'(s)$ does not vanish on W where $k = \dim L$. On the dense open set $U = \{s \in W : D(s) \neq 0\}$ there is a unique solution $v'(s)$ for the $k \times k$ sub-system and by Cramer's rule $v'(s)$ is meromorphic on W . Clearly, $\lambda(s)(v'(s))$ is the unique solution of $\Xi(s)$ on U and in particular $\lambda(s)(v'(s)) = v(s)$ on $S_0 \cap W$. \square

It remains to show that the system defined by $\Delta f = (1/4 - s^2)f$ is of locally finite type. This is a technical strengthening of Harish-Chandra's finiteness theorem. It can be proved along the same lines.

Applications: Computation of the volume of the fundamental domain (Langlands, Boulder '65)



Naively, we can try to compute $\text{vol}(\Gamma \backslash \mathcal{H})$ by computing

$$I(s) = \int_{\Gamma \backslash \mathcal{H}} E(z; s) dz$$

and taking residue at $s = \frac{1}{2}$. The problem is that $E(z; s) \notin L^1(\Gamma \backslash \mathcal{H})$ in the range of convergence. On the other hand $E(z; s) \in L^1(\Gamma \backslash \mathcal{H})$ if $|\text{Re}(s)| < \frac{1}{2}$. However, we will soon see that $I(s) \equiv 0$. (We cannot take the limit inside the integral because of the non-compactness of the domain.)

Instead we take for any $f \in C_c^\infty(\mathbb{R}_{>0})$ the wave packet

$$\theta_f(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} f(\text{Im } \gamma z)$$

This is a finite sum, and θ_f is compactly supported in $\Gamma \backslash \mathcal{H}$. By Mellin inversion,

$$f(y) = \int_{\text{Re}(s)=s_0} \hat{f}(s) y^{s+\frac{1}{2}} ds$$

for any s_0 where \hat{f} is the Mellin transform of f

$$\hat{f}(s) = \int_{\mathbb{R}_{>0}} f(y) y^{-s-\frac{1}{2}} \frac{dy}{y}.$$

(It is an entire function of Paley-Wiener type.)

Thus,

$$\begin{aligned} \theta_f(z) &= \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \int_{\operatorname{Re} s = s_0} \hat{f}(s) (\operatorname{Im} \gamma z)^{s+\frac{1}{2}} \\ &= \int_{\operatorname{Re} s = s_0} \hat{f}(s) E(z; s) \end{aligned}$$

provided that $s_0 > \frac{1}{2}$. We can compute

$$I = \int_{\Gamma \setminus \mathcal{H}} \theta_f(z) dz$$

in two different ways. On the one hand we can shift the contour to $\operatorname{Re} s = 0$ acquiring a residue at $s = \frac{1}{2}$ to get

$$I = \operatorname{vol}(\Gamma \setminus H) \frac{6}{\pi} \hat{f}\left(\frac{1}{2}\right) + \int_{\operatorname{Re}(s)=0} \hat{f}(s) I(s) dz$$

On the other hand we can compute I directly using the definition of θ_f . Unfolding the inte-

gral and the sum we get

$$I = \int_{\Gamma_\infty \backslash \mathcal{H}} f(\text{Im } z) dz = \int_{\mathbb{R}_{>0}} f(y) \frac{dy}{y^2} = \hat{f}\left(\frac{1}{2}\right)$$

Comparing the two formulae (as distributions in \hat{f}) we infer that

$$\text{vol}(\Gamma \backslash \mathcal{H}) = 2 \text{vol}(\bar{\Gamma} \backslash \mathcal{H}) = \frac{\pi}{6}$$

and $I(s) \equiv 0$. We used the following Lemma
Lemma 2. *Suppose that $I(t)$ is bounded and*

$$\int \hat{f}(it) I(t) dt = a \hat{f}(1)$$

for all $f \in \mathcal{C}_c^\infty(\mathbb{R}_{>0})$. Then $I \equiv a = 0$.

Proof. By taking $f_1 = yf' - f$ we have $\hat{f}_1 = (s - 1)\hat{f}$ and therefore $\int \hat{f}(it) I_1(t) dt = 0$ for $I_1 = (it - 1)I$. Since this is true for all f , $I_1 \equiv 0$. Therefore $I \equiv 0$. □

Remark: Using this method Langlands computed $\text{vol}(G(\mathbb{Z}) \backslash G(\mathbb{R}))$ for any semisimple Cheval-

ley group. For non-split groups this was completed by Kottwitz using the trace formula, leading to the solution of a conjecture of Weil's.

Prime Number Theorem (with remainder)
(Sarnak, Shalika 60th birthday volume)

Truncated Eisenstein series: for z in the Siegel domain set

$$\Lambda^T E(z; s) = \begin{cases} E(z; s) & y \leq T, \\ E(z; s) - a_0(y, s) & y > T. \end{cases}$$

It is rapidly decreasing at the cusp. Maass-Selberg relations:

$$\|\Lambda^T E(z; it)\|_2^2 = 2 \log T - \frac{\phi'}{\phi}(it) + \frac{\overline{\phi}(it) T^{2it} - \phi(it) T^{-2it}}{2it}.$$

where $\phi(s) = \frac{\zeta^*(2s)}{\zeta^*(2s+1)}$. Note that $|\phi(it)| = 1$ and

$$\begin{aligned} \frac{\phi'}{\phi}(it) &= \operatorname{Re} \frac{\zeta^{*'}}{\zeta^*}(1 + 2it) \\ &= \frac{\zeta'}{\zeta}(1 + 2it) + \frac{\Gamma'}{\Gamma}\left(\frac{1}{2} + it\right) + \frac{1}{2} \log \pi. \end{aligned}$$

Thus, for T fixed, and $t \geq 2$

$$\begin{aligned} &\|\zeta(1 + 2it) \wedge^\top E(z; it)\|_2^2 \leq \\ &|\zeta(1 + 2it)| \left(|\zeta(1 + 2it)| + |\zeta'(1 + 2it)| + \frac{\Gamma'}{\Gamma}\left(\frac{1}{2} + it\right) + 3 \right) \end{aligned}$$

By standard *upper* bounds for $\zeta(1 + it)$ and $\zeta'(1 + it)$ this is majorized by

$$|\zeta(1 + 2it)| (\log t)^2$$

OTOH

$$\begin{aligned} &\|\zeta(1 + 2it) \wedge^\top E(z; it)\|_2^2 \geq \\ &\int_1^\infty \int_0^1 \left| \zeta(1 + 2it) \wedge^\top E(x + iy; it) \right|^2 \frac{dx dy}{y^2}. \end{aligned}$$

By Bessel's inequality

$$\geq \sum_{m=1}^{\infty} \int_1^{\infty} \left| \frac{K_{it}(2\pi |m| y) \sigma_{-2it}(m)}{\Gamma(\frac{1}{2} + it)} \right|^2 \frac{dy}{y}$$

Taking only $m = 1$ and comparing the two inequalities we get

$$\int_1^{\infty} \left| \frac{K_{it}(2\pi y)}{\Gamma(\frac{1}{2} + it)} \right|^2 \frac{dy}{y} \ll |\zeta(1 + 2it)| (\log t)^2$$

Using the precise asymptotic for the Bessel function in the regime $t/8 < y < t/4$, LHS $\gg \frac{1}{t}$ and therefore

$$|\zeta(1 + 2it)| \gg \frac{1}{t(\log t)^2}$$

In fact, we would have more precisely

$$\frac{1}{t} \sum_{m \leq t/8} |\sigma_{-2it}(m)|^2 \ll |\zeta(1 + 2it)| (\log t)^2.$$

The fact that for p prime

$$\left| \sigma_{-2it}(p) - \sigma_{-2it}(p^2) \right| = 1$$

guarantees that

$$|\sigma_{-2it}(p)|^2 + |\sigma_{-2it}(p^2)|^2 \geq \frac{1}{2}$$

so that at least

$$\sum_{m \leq t/8} |\sigma_{-2it}(m)|^2 \geq \frac{1}{2} \sum_{p \leq \sqrt{t/8}: p \text{ prime}} \gg \sqrt{t}/\log t$$

by Chebyshev. This gives

$$|\zeta(1 + 2it)| \gg \frac{1}{\sqrt{t}(\log t)^3}$$

By refining the argument one can get

$$|\zeta(1 + 2it)| \gg \frac{1}{(\log t)^3}$$

which gives a zero-free region which is almost as good as the standard one (à la de la Vallée Poussin).

Gauss class number problem

Gauss conjectured that $h(D) \rightarrow \infty$ as $D \rightarrow -\infty$ and gave a table for the D 's with small class number.

It was known to Hecke and Landau in the 1920's that under GRH, $h(D) \gg \sqrt{D}/\log D$.
 Deuring ('33) If RH is false, then $h(D) = 1$ for only finitely many $D < 0$.

$$\sum_{z \in \Lambda_D} E(z; s) = \zeta_{Q(\sqrt{D})}(s + \frac{1}{2}) = \zeta(s + \frac{1}{2})L(s + \frac{1}{2}, \chi_D)$$

Suppose that $\zeta(s_0 + \frac{1}{2}) = 0$ with $\text{Re}(s_0) > 0$. Then LHS is zero at s_0 for all D . However, if $h(D) = 1$ then LHS is just $E(\frac{\delta + \sqrt{D}}{2}; s_0)$ with $\delta = 0, 1, \delta \equiv D \pmod{4}$. OTOH,

$$\frac{E(\frac{\delta + \sqrt{D}}{2}; s_0)}{\sqrt{|D|}} = |D|^{s_0} + \phi(s_0) |D|^{-s_0} + O(|D|^{-N})$$

for all $N > 0$. Clearly the first term on the RHS is dominant since $\text{Re}(s_0) > 0$, and therefore LHS cannot vanish.

Remark: Deuring's idea was quickly generalized by Heilbronn and Siegel to show Gauss' conjecture under $\neg GRH$, (and therefore solving

it, albeit non-effectively). The best effective lower bound known is roughly $\log D$ (Goldfeld, Gross-Zagier). Interestingly enough it relies on a high order zero for an L -function (which is “not very far” from Deuring’s point of departure).

Spectral decomposition.

Let

$$L^2(\Gamma \backslash \mathcal{H}) = L_{disc}^2(\Gamma \backslash \mathcal{H}) \oplus L_{cont}^2(\Gamma \backslash \mathcal{H})$$

be the spectral decomposition of Δ into a discrete and continuous part respectively. Let $L_{cusp}^2(\Gamma \backslash \mathcal{H})$ be the space of cusp forms, i.e. those f such that

$$\int_0^1 f(x + iy) dx = 0 \text{ for almost all } y.$$

A-priori, it is not clear that $L_{cusp}^2(\Gamma \backslash \mathcal{H}) \neq 0$! At any rate, it is a fact that Δ decomposes discretely on $L_{cusp}^2(\Gamma \backslash \mathcal{H})$.

Theorem 1. $L^2_{disc}(\Gamma \backslash \mathcal{H}) = L^2_{cusp}(\Gamma \backslash \mathcal{H}) \oplus \mathbb{C} \cdot 1$

The map $L^2(\mathbb{R}_{\geq 0}) \rightarrow L^2(\Gamma \backslash \mathcal{H})$ given by

$$f \mapsto Ef = \int f(it)E(z; it) dt$$

is an isometry onto $L^2_{cont}(\Gamma \backslash \mathcal{H})$ and

$$\Delta(Ef) = E\left(\frac{1}{4} - t^2\right)f$$

Alternatively, any $f \in L^2(\Gamma \backslash \mathcal{H})$ has a decomposition

$$f(z) = \sum_j (f, u_j)u_j(z) + \frac{1}{4\pi} \int_{-\infty}^{\infty} (f, E(\cdot; it))E(z; it) dt$$

in terms of eigenfunctions of Δ . The first sum is taken over an orthonormal basis of the discrete part. Equivalently,

$$\|f\|_2^2 = \sum_j |(f, u_j)|^2 + \frac{1}{4\pi} \int_{-\infty}^{\infty} |(f, E(\cdot; it))|^2 dt$$

First note:

$$(\theta_f, \varphi)_{\Gamma \backslash \mathcal{H}} = (f, \varphi_U)_{\mathbb{R}_{>0}}$$

for any φ on $\Gamma \setminus \mathcal{H}$. Therefore the space spanned by θ_f is the orthogonal complement of $L^2_{cusp}(\Gamma \setminus \mathcal{H})$. Now,

$(\theta_{f_1}, \theta_{f_2})_{\Gamma \setminus \mathcal{H}} = ((\theta_{f_1})_U, f_2)_{\mathbb{R}_{>0}} = (f_1 + Mf_1, f_2)_{\mathbb{R}_{>0}}$ where $Mf(y) = \int_{\mathbb{R}} f(y/(x^2 + y^2)) dx$. We have

$$\widehat{M}f(s) = \phi(s)\widehat{f}(-s)$$

By isometry of Fourier transform

$$(f_1, f_2) = \int_{\text{Re } s = s_0} \widehat{f}_1(s) \overline{\widehat{f}_2(-\bar{s})} ds$$

We get

$$\begin{aligned} (\theta_{f_1}, \theta_{f_2}) &= \int_{\text{Re } s = s_0} \widehat{f}_1(s) \overline{\widehat{f}_2(-\bar{s})} ds + \\ &\quad \int_{\text{Re } s = s_0 \gg 0} \widehat{f}_1(s) \phi(s) \overline{\widehat{f}_2(\bar{s})} ds = \\ &\quad \int_{\text{Re } s = 0} \widehat{f}_1(s) \overline{\widehat{f}_2(s)} ds + \\ &\quad \int_{\text{Re } s = 0} \widehat{f}_1(s) \phi(s) \overline{\widehat{f}_2(-s)} ds + \text{Res}_{s=\frac{1}{2}} \phi(s) \cdot \widehat{f}_1\left(\frac{1}{2}\right) \overline{\widehat{f}_2\left(\frac{1}{2}\right)} \end{aligned}$$

On the other hand, $(\theta_{f_1}, \theta_{f_2})$, i.e.

$$\left(\int_{\text{Re } s = s_0 \gg 0} \widehat{f}_1(s) E(\cdot, s), \int_{\text{Re } s = s_0 \gg 0} \widehat{f}_2(s) E(\cdot, s) \right)$$

can be written, again by shift of contour, as

$$\begin{aligned} & \left(\int_{\operatorname{Re} s=0} \hat{f}_1(s) E(\cdot, s) ds, \int_{\operatorname{Re} s=0} \hat{f}_2(s) E(\cdot, s) ds \right) + \\ & \quad \overline{\hat{f}_2(s_0)} \left(\int_{\operatorname{Re} s=0} \hat{f}_1(s) E(\cdot, s) ds, \operatorname{Res}_{s_0} E \right) + \\ & \quad \hat{f}_1\left(\frac{1}{2}\right) \left(\operatorname{Res}_{s=\frac{1}{2}} E, \int_{\operatorname{Re} s=0} \hat{f}_2(s) E(\cdot, s) ds \right) + \\ & \quad \hat{f}_1\left(\frac{1}{2}\right) \overline{\hat{f}_2(s_0)} \left(\operatorname{Res}_{s=\frac{1}{2}} E, \operatorname{Res}_{s=\frac{1}{2}} E \right). \end{aligned}$$

Comparing, we get

$$\left(\operatorname{Res}_{s=\frac{1}{2}} E(\cdot, s), \operatorname{Res}_{s=\frac{1}{2}} E(\cdot, s) \right) = \operatorname{Res}_{s=\frac{1}{2}} \phi(s),$$

$$\left(\operatorname{Res}_{s=\frac{1}{2}} E(s) ds, \int_{\operatorname{Re} s=0} \hat{f}(s) E(\cdot, s) ds \right) = 0$$

and

$$\begin{aligned} & \left(\int_{\operatorname{Re} s=0} \hat{f}_1(s) E(\cdot, s) ds, \int_{\operatorname{Re} s=0} \hat{f}_2(s) E(\cdot, s) ds \right) = \\ & \int_{\operatorname{Re} s=0} (\hat{f}_1(s) + \phi(s) \hat{f}_1(-s)) \overline{(\hat{f}_2(s) + \phi(s) \hat{f}_2(-s))} \end{aligned}$$

This gives rise to spectral decomposition.

Connection with the holomorphic Eisenstein series

pass to group setup: consider

$$E(\varphi, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \varphi(\gamma g) y(\gamma g i)^{s + \frac{1}{2}}$$

where $\varphi : B \backslash G \rightarrow \mathbb{C}$ where $B = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$. Previously we used $\varphi \equiv 1$ which gives rise to function $E(gi; s)$.

Now we get an intertwining map from $I(s) = \text{Ind}_B^G \left(\begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix} \mapsto \left| \frac{t_1}{t_2} \right|^s \right)$ to the space of automorphic forms on $\Gamma \backslash G$.

For example, taking

$$\varphi_k \left(\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \right) = e^{i\theta k}$$

and $s = k - \frac{1}{2}$. Then for $z = gi$

$$\begin{aligned} G_{2k}(z) &= \zeta(2k) \left(\frac{ci + d}{|ci + d|} \right)^k E(g, \varphi_k, k - \frac{1}{2}) \\ &= \sum_{(m,n) \neq 0} (mz + n)^{-2k} \end{aligned}$$

is the holomorphic Eisenstein series. It has Fourier expansion

$$2\zeta(2k) \left(1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n \right) \quad q = e^{2\pi iz}$$

Note that $I(k - \frac{1}{2})$ is reducible:

$$0 \rightarrow F_{2k-1} \rightarrow I(k - \frac{1}{2}) \rightarrow D_{2k-1} \rightarrow 0$$

where F_l is the l -dimensional irreducible representation of $SL_2(\mathbb{R})$ and D_l is the discrete series representation. φ_k is the lowest K -type in D_{2k-1} .

Kronecker limit formula

$$E(z; s) = \frac{c_0}{s - \frac{1}{2}} + c_1 \log(y^6 |\Delta(z)|) + c_2 + O(s - \frac{1}{2})$$

for certain constants c_0, c_1, c_2 where

$$\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} \quad q = e^{2\pi iz}$$

Spectral theory for GL_2 - adelic version.

Let R be the right regular representation of $G(\mathbb{A})$ on $L^2(G(F)\backslash G(\mathbb{A}))$, i.e. $R(g)\varphi(x) = \varphi(xg)$ for $\varphi \in L^2(G(F)\backslash G(\mathbb{A}))$. For any $f \in C_c^\infty(G(\mathbb{A}))$ let $R(f)$ be the operator $\int_{G(\mathbb{A})} f(g)R(g) dg$, that is

$$R(f)\varphi(x) = \int_{G(\mathbb{A})} f(g)\varphi(xg) dg.$$

Then

$$\begin{aligned} R(f)\varphi(x) &= \int_{G(\mathbb{A})} f(x^{-1}g)\varphi(g) dg = \\ &= \int_{G(F)\backslash G(\mathbb{A})} \sum_{\gamma \in G(F)} f(x^{-1}\gamma y)\varphi(\gamma y) dg \\ &= \int_{G(F)\backslash G(\mathbb{A})} K_f(x, y)\varphi(y) dy \end{aligned}$$

i.e., $R(f)$ is an integral operator on $L^2(G(F)\backslash G(\mathbb{A}))$ with kernel

$$K_f(x, y) = \sum_{\gamma \in G(F)} f(x^{-1}\gamma y)$$

The spectral theory for PGL_2 gives

$$K_f(x, y) = K_f^{cusp}(x, y) + K_f^{res}(x, y) + K_f^{cont}(x, y)$$

where

$$K_f^{cusp} = \sum_{\{\varphi\}} R(f)\varphi(x)\overline{\varphi(y)}$$

the sum is taken over an orthonormal basis of cusp forms;

$$K_f^{res}(x, y) = \sum_{\substack{\chi: F^*\backslash \mathbb{I}_F \rightarrow \mathbb{C}^* \\ \chi^2=1}} \text{vol}(G(F)\backslash G(\mathbb{A}))^{-1} \int_{G(\mathbb{A})} f(g)\chi(\det g) dg \cdot \chi(\det xy^{-1})$$

and

$$K_f^{cont}(x, y) = \sum_{\chi: F^* \setminus \mathbb{I}_F^1 \rightarrow \mathbb{C}^*} \sum_{\{\varphi\}} \int_{-\infty}^{\infty} E(x, I(f, \chi, it)\varphi, it) \overline{E(y, \varphi, it)} dt$$

where $\{\varphi\}$ is an orthonormal basis of the space

$$I(\chi) = \left\{ \varphi : G(\mathbb{A}) \rightarrow \mathbb{C} \mid \varphi\left(\begin{pmatrix} t & * \\ 0 & 1 \end{pmatrix} g\right) = \chi(t) |t|^{\frac{1}{2}} \varphi(g) \right\}$$

with

$$(\varphi_1, \varphi_2) = \int_{\mathbb{R}_{>0} T(F)U(\mathbb{A}) \backslash G(\mathbb{A})} \varphi_1(g) \overline{\varphi_2(g)} dg$$

and for

$$\varphi_s\left(\begin{pmatrix} t & * \\ 0 & 1 \end{pmatrix} k\right) = |t|^s \varphi(k),$$

$$I(g, \chi, s)\varphi(\cdot) = (\varphi_s(\cdot g))_{-s}$$

Mirabolic Eisenstein series for GL_n . Let V be an n -dimensional space over \mathbb{Q} and let \tilde{V} be

the dual space. For $\Phi \in \mathcal{S}(V(\mathbb{A}))$ set

$$E_{\Phi}^V(g, s) = |\det g|^{\frac{s}{n} + \frac{1}{2}} \int_0^{\infty} \sum_{v \in V(\mathbb{Q}) \setminus \{0\}} \Phi_g(tv) |t|^{s+n/2} \frac{dt}{t}$$

where $\Phi_g(\cdot) = \Phi(\cdot g)$, $g \in GL(V(\mathbb{A}))$ acting on the right on $V(\mathbb{A})$. This is the Mellin transform of $\Theta_{\Phi_g}^* = \Theta_{\Phi_g} - \Phi(0)$ where

$$\Theta_{\Phi}(t) = \sum_{v \in V(\mathbb{Q})} \Phi(tv) \quad t \in \mathbb{R}_{>0}.$$

By Poisson summation formula

$$\Theta_{\Phi}(t) = t^{-n} \Theta_{\hat{\Phi}}(t^{-1})$$

where $\hat{\Phi} \in \mathcal{S}(\tilde{V}(\mathbb{A}))$ is given by

$$\hat{\Phi}(\tilde{v}) = \int_{V(\mathbb{A})} \Phi(v) \psi((\tilde{v}, v)) \, dv \quad \tilde{v} \in \tilde{V}(\mathbb{A})$$

where ψ is a fixed non-trivial character of $\mathbb{Q} \setminus \mathbb{A}$.

Also,

$$\widehat{\Phi}_g = |\det g|^{-1} \hat{\Phi}_{g^*}$$

where $(\tilde{v}g^*, v) = (\tilde{v}, vg^{-1})$. By Tate's thesis,

$$\begin{aligned}
 E_{\Phi}^V(g, s) &= \\
 &|\det g|^{\frac{s}{n} + \frac{1}{2}} \left(\int_1^{\infty} \Theta_{\Phi_g}^*(t) t^{s+n/2} \frac{dt}{t} - \frac{\Phi(0)}{s+n/2} \right) + \\
 &|\det g^*|^{\frac{1}{2} - \frac{s}{n}} \left(\int_1^{\infty} \Theta_{\hat{\Phi}_{g^*}}^*(t) t^{n/2-s} \frac{dt}{t} + \frac{\hat{\Phi}(0)}{s-n/2} \right) \\
 &= E_{\hat{\Phi}}^{\tilde{V}}(g^*, -s).
 \end{aligned}$$

Note: Let $V = K$ a field extension K of degree n . Then K^* is a torus in GL_n and for any Hecke character χ of \mathbb{I}_K we have

$$\int_{K^* \setminus \mathbb{I}_K^1} E_{\Phi}^V(t, s) \chi(t) dt = \int_{K^* \setminus \mathbb{I}_K} \Theta_{\Phi}^*(t) \chi(t) |t|^{\frac{s}{n} + \frac{1}{2}} dt$$

which is Tate's integral for $L(\frac{s}{n} + \frac{1}{2}, \chi)$

More generally, starting with a cusp form ϕ on $GL_n(F) \setminus GL_n(\mathbb{A})$ we can construct following

Jacquet-Shalika, for each $\Phi \in \mathcal{S}(M_{n \times (n+1)}(\mathbb{A}))$

$$E(g; \Phi, \phi, s) = |\det g|^{ns} \int_{GL_n(F) \backslash GL_n(\mathbb{A})} \sum_{\substack{\eta \in M_{n \times (n+1)}(F) \\ \text{rk } \eta = n}} \Phi(x^{-1} \eta g) \phi(x) |\det x|^{-(n+1)s} dx$$

As in Godement-Jacquet, write this as

$$\begin{aligned} & |\det g|^{ns} \int_{|\det x| \geq 1} \theta[x^{-1} \Phi g] \phi(x) |\det x|^{-(n+1)s} dx + \\ & |\det g^*|^{n(1-s)} \int_{|\det x| \geq 1} \theta[x^{-1} \hat{\Phi}_{g^*}] \phi^*(x) |\det x|^{-(n+1)(1-s)} dx \\ & \qquad \qquad \qquad = E(g^*; \hat{\Phi}, \phi^*, 1-s) \end{aligned}$$

where $g \in GL_{n+1}(\mathbb{A})$, $\phi^*(x^*) = \phi(x)$, $x \Phi_g(y) = \Phi(xyg)$

$$\hat{\Phi}(x) = \int_{M_{n \times (n+1)}(\mathbb{A})} \Phi(y) \psi(\text{tr}(y \cdot {}^t x)) dy$$

and $\theta[\Phi] = \sum_{\substack{\xi \in M_{n \times (n+1)}(F) \\ \text{rk } \xi = n}} \Phi(\xi)$.