Automorphic forms with spikes

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X - compact Riemann surface, Δ - Laplace-Beltrami operator, φ eigenfunction $\Delta\varphi+\lambda^2\varphi=0$ (always with $\|\varphi\|_2=1$). Then

$$\|\varphi\|_\infty^2 \leq c_X\lambda \text{ (Seeger-Sogge).}$$

This is sharp for $X=S^2$.

For an arithmetic quotient $X=\Gamma\backslash\mathbb{H}^2$ of the upper half-plane $\mathbb{H}^2=\{z\in\mathbb{C} : \Im z>0\}$, one can improve if φ is a Hecke eigenform:

$$\|\varphi\|_\infty^2 \ll \lambda^{5/6+\epsilon} \text{ (Iwaniec-Sarnak)}$$

for every $\epsilon>0$. Moreover,

Conjecture (Iwaniec-Sarnak). *For all* $\epsilon>0$

$$\|\varphi\|_\infty^2 \ll \lambda^\epsilon.$$

(Recall: local Weyl law. For any $x\in X$

$$\sum_{\lambda\leq T}|\varphi(x)|^2 \sim c_X T^2$$

so that on average $\varphi(x)$ is small.)

(For non-compact arithmetic quotients like $X = SL_2(\mathbb{Z})\backslash\mathbb{H}^2$ the conjecture would be

$$\|\varphi\|_{\infty,K} \leq c_{\varepsilon,K}\lambda^{\epsilon}$$

for any $K \subset X$ compact.)

This conjecture is very strong. It implies the multiplicity conjecture $m(\lambda) \ll \lambda^{\epsilon}$.

Perspective: Classical Eisenstein series

$$E(z,s) = (\Im z)^{s+\frac{1}{2}} \sum_{(m,n)\in\mathbb{Z}^2\backslash 0} |mz+n|^{-2s-1}$$

We have

$$E(\sqrt{-1}, \mathrm{i}t) = \zeta_{\mathbb{Q}(\sqrt{-1})}(\frac{1}{2} + \mathrm{i}t)$$

The analogue of the conjecture for Eisenstein series implies Lindelöf for $\zeta_{\mathbb{Q}(\sqrt{-1})}$. Trivial bound confirms with convexity bound.

For cusp forms: Waldspurger's formula

$$\left|\phi(\sqrt{-1})\right|^2 = \frac{L(\frac{1}{2},\pi)L(\frac{1}{2},\pi\otimes\chi_{-4})}{L(1,\pi,\mathrm{sym}^2)}$$

Conjecture implies Lindelöf for $L(\frac{1}{2}, \pi)$. More generally, for any $d < 0$, $\|\varphi\|_2 = 1$

$$\left| \sum_{x \in \Lambda_d} \varphi(x) \right|^2 = |d|^{\frac{1}{2}} \frac{L(\frac{1}{2}, \pi) L(\frac{1}{2}, \pi \otimes \chi_d)}{L(1, \pi, \mathrm{sym}^2)}$$

$\Lambda_d = \{\frac{-b+\sqrt{d}}{2a} : a, b, c \in \mathbb{Z}, b^2 - 4ac = d\}/SL_2(\mathbb{Z})$.

(This underlines the connection between equidistribution of CM-points and a subconvexity bound $L(\frac{1}{2}, \pi \otimes \chi_d) \ll |d|^{1/2-\delta}$ for some $\delta > 0$. Iwaniec, Duke)

In fact, for any character χ of the class group of $\mathbb{Q}(\sqrt{d})$

$$\left|\sum_{x\in\Lambda_d}\chi(x)\varphi(x)\right|^2 = |d|^{\frac{1}{2}}\,\frac{L(\frac{1}{2},\pi\times\theta(\chi))}{L(1,\pi,\mathrm{sym}^2)}$$

Thus,

$$\frac{1}{|d|^{\frac{1}{2}}}\sum_{x\in\Lambda_d}|\varphi(x)|^2 = \frac{\sum_\chi L(\frac{1}{2},\pi\times\theta(\chi))}{h(d)L(1,\pi,\mathrm{sym}^2)}$$

So we can get

$$\frac{1}{h(d)}\sum_\chi L(\frac{1}{2},\pi\times\theta(\chi)) \le |d|^\varepsilon\,\lambda^{1-\delta}$$

but we cannot get pointwise bounds on φ from bounds on the L-function.

What about lower bounds?

For example $\zeta(\frac{1}{2}+\mathrm{i}t)$. Moments

$$M_k(T) = \frac{1}{T}\int_1^T \left|\zeta(\frac{1}{2}+\mathrm{i}t)\right|^{2k}$$

Conjecture: $M_k(T) \sim c_k(\log T)^{k^2}$. Upper bounds of the right order of magnitude are not known beyond $k = 2$. However it is known that $M_k(T) \gg (\log T)^{k^2}$ (Ramachandra). Therefore for any k, $\left|\zeta(\frac{1}{2}+\mathrm{i}t)\right| \geq (\log t)^k$ for arbitrarily large t. Recently Rudnick-Soundararajan developed a general (and simple!) method to obtain such lower bounds for many families. With these ideas, Sarnak showed that for any k

$$\|\varphi\|_\infty \geq (\log \lambda)^k$$

for many φ's.

In fact, roughly speaking, Sarnak shows that Maass forms take large values on CM-points. (A pre-trace formula approach; analogous to Selberg's Λ^2 sieve).

Higher dimension

$X_\Gamma = \Gamma\backslash G/K$ a locally symmetric space $d = \dim G/K = \dim X$, G semisimple, K maximal compact, Γ lattice of G. A – algebra of invariant differential operators on G/K (a polynomial ring on $r = \operatorname{rk} G$ variables).

If φ is a cusp form, eigenfunction of A with Harish-Chandra parameter λ (an r-dimensional parameter) then

$$\|\varphi\|_\infty^2 \leq c_X \|\lambda\|^{d-r}$$

More precisely, $\lambda \in X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}$, T – maximal split torus

$$\|\varphi\|_\infty^2 \leq c_X \mu(\lambda) \text{ (Sarnak-Venkatesh)}$$

$$\text{where } \mu(\lambda) = \prod_{\alpha \in \Phi^+} \left(\left|\left\langle \lambda, \alpha^\vee \right\rangle\right| + 1\right)^{m(\alpha)}$$

Φ^+ is the set of positive roots. (This is sharper than the previous bound if λ is close to the walls.)

In many arithmetic cases when φ is Hecke eigenfunction, one can improve

$$\|\varphi\|_\infty^2 \leq c_X \mu(\lambda)^\delta \text{ (Sarnak-Venkatesh)}$$

with $\delta < 1$. However, in general we cannot expect to be able to take δ arbitrarily small.

Example (Rudnick-Sarnak): An arithmetic quotient of the hyperbolic 3-space $SL_2(\mathbb{Z}[\sqrt{-1}])\backslash\mathbb{H}^3 = SL_2(\mathbb{Z}[\sqrt{-1}])\backslash SL_2(\mathbb{C})/SU(2)$.
It turns out that $\varphi(I) = 0$ unless φ is invariant under Galois conjugation, i.e. unless ϕ is a theta lift from $\Gamma_0(4)\backslash\mathbb{H}^2$. Therefore, although

$$\sum_{\lambda \leq T} |\varphi(I)|^2 \sim c_X T^3$$

only T^2 summands contribute (corresponding to Maass forms on $\Gamma_0(4)\backslash\mathbb{H}^2$ with the same parameter). Thus, on average $|\varphi(I)|^2 \sim \lambda$ for the theta lifts, so that $\|\varphi\|_\infty^2 \geq \lambda$, at least on average. (The upper bound is $\|\varphi\|_\infty^2 \leq \lambda^2$.)

The relative trace formula

E/F – quadratic extension of number fields $G = GL_n(E)$; $H = U_n \subset G$ – a unitary group.

Theorem. *The following are equivalent for a cuspidal representation* Π *of* $G(\mathbb{A})$.

1. Π *is a base change from* GL_n/F.

2. $\bar{\Pi} = \Pi$.

3. $\int_{H(F)\backslash H(\mathbb{A})} \varphi(h)\ dh$ *does not vanish identically on the space of* π.

1 implies 2: easy.

3 implies 2: rather easy (an argument of Harder-Langlands-Rapoport.)

2 implies 1: Arthur-Clozel (twisted trace formula).

1 implies 3: Jacquet (relative trace formula).

The idea: for $f \in C_c^\infty(GL_n(\mathbb{A}))$ let

$$K_f(x, y) = \sum_{\gamma \in GL_n(F)} f(x^{-1}\gamma y)$$

be the automorphic kernel. Similarly, for $\Phi \in C_c^\infty(Herm_n(\mathbb{A}))$ let

$$K_\Phi(x) = \sum_{\gamma \in Herm_n(F)} \Phi(\bar{x}^t \gamma x), \quad x \in GL_n(\mathbb{A}_E).$$

Now, compare the *Kuznetsov trace formula*

$$\iint_{(N_n(F)\backslash N_n(\mathbb{A}_F))^2} K_f(u_1, u_2)\psi(u_1 u_2^{-1})\ du_1\ du_2,$$

where ψ is a non-degenerate character on the group N_n of upper triangular unipotent matrices, with its relative analogue

$$\int_{N_n(E)\backslash N_n(\mathbb{A}_E)} K_\Phi(u)\psi(\mathrm{tr}\, u)\ du$$

for appropriate pairs of matching $f = \prod f_v \leftrightarrow \Phi = \prod_v \Phi_v$.

How to compare?

The Kuznetsov trace formula decomposes as a sum over *relevant* double cosets $N_n\backslash GL_n/N_n$ (not killed by ψ) of orbital integrals, which are products of local distributions. Similarly, its relative analogue is a sum over relevant $N_n(E)$-orbits on $Herm_n$ of orbital integrals. The matching conditions are equality of these orbital integrals (locally), namely:

$$\int_{N_n(F_v)\times N_n(F_v)} f_v(u_1^t a u_2)\psi(u_1 u_2)\ du_1\ du_2 \\ = \eta_v(a)\int_{N_n(E_v)} \Phi_v(\bar{u}^t a u)\psi(\mathrm{tr}\, u)\ du$$

for any diagonal $a \in GL_n(F_v)$ for a certain quadratic character $\eta = \otimes\eta_v$.

Need to show: existence of enough matching functions, and compatibility with Hecke Algebra base change homomorphism (Fundamental Lemma). (Jacquet)

Spectral decomposition. This is easy in the Kuznetsov side, but more tricky in the relative side (L). Difficulties: discrete terms coming from continuous spectrum; absolute convergence.

At this stage we get the following spectral identity. Let $\{\xi\}$ be a set of representatives of equivalence classes of Hermitian forms. Let $\{f_\xi\} \subset C_c^\infty(GL_n(\mathbb{A}_E))$ be such that

$$\Phi(\bar{g}^t \xi g) = \int_{H_\xi(\mathbb{A})} f_\xi(hg)\ dh$$

where H_ξ is the unitary group defined by ξ.

$$\sum_\xi \sum_{\phi \in \mathrm{ob}(\Pi)} \int_{H_\xi(F)\backslash H_\xi(\mathbb{A})} \Pi(f_\xi)\phi(h)\ dh \cdot \overline{\int_{N_n(E)\backslash N_n(\mathbb{A}_E)} \phi(u)\psi(\mathrm{tr}\, u)\ du}$$

$$= \sum_{\pi : \mathrm{bc}(\pi) = \Pi} \sum_{\varphi \in \mathrm{ob}(\pi)} \int_{N_n(F)\backslash N_n(\mathbb{A})} \pi(f)\varphi(u)\psi(u)\, du \cdot \overline{\int_{N_n(F)\backslash N_n(\mathbb{A})} \varphi(u)\psi(u)\ du}$$

In particular, if the left-hand side is non-zero then $\Pi = \mathrm{bc}(\pi)$ for some π.

In fact, the relative trace formula give more information. The Rankin-Selberg integral yields

$$\begin{aligned}(\varphi_1, \varphi_2)_{GL_n(F)\backslash GL_n(\mathbb{A})} \\ = \operatorname{res}_{s=1} L^S(s, \pi \otimes \tilde{\pi}) \prod_{v \in S} [W_v^1, W_v^2]\end{aligned}$$

where $W^i(g) = \prod_v W_v^i$ is the Whittaker function of φ_i

$$W^i(g) = \int_{N_n(F)\backslash N_n(\mathbb{A})} \varphi_i(ug)\psi(u)\ du$$

The right-hand side factorizes into a product of local distributions.

This forces a factorization of the left-hand side. (This is non-trivial since locally, there is NO multiplicity one for invariant H_ξ- functionals!) Also, one can obtain in principle a formula for the unitary period.

$X = GL_n(\mathcal{O}_D) \backslash GL_n(\mathbb{C})/U_n$, $\mathcal{O}_D =$ integers of $Q(\sqrt{D})$, $D < 0$. φ cusp L^2-normalized Hecke eigenform on X. Let

$$\Lambda = GL_n(\mathcal{O}_D) \backslash GL_n(Q(\sqrt{D}))U_n/U_n \subset X$$

the genus of the standard Hermitian form $\|\cdot\|^2$. Then $\sum_{x \in \Lambda} \varphi(x) = 0$ unless φ is a base change from a cusp form on $\Gamma_0(D) \backslash GL_n(\mathbb{R})/O_n$. In this case, if π is the corresponding representation then (L-Offen)

$$\left| \sum_{x \in \Lambda} \varphi(x) \right|^2 = *(\text{local factors}) \frac{L(1, \pi \times \tilde{\pi} \times \chi_D)}{\text{res}_{s=1} L(s, \pi \times \tilde{\pi})}$$

χ_D - quadratic character of conductor D.

Remarks: The result is more general for E − CM field, F − max. totally real subfield. The local factors are 1 in the case where E/F is unramified at all finite places. (This excludes the previous case $F = \mathbb{Q}$.) The local factors are related to spherical functions which were computed by Yumiko Hironaka. Another crucial ingredient is an identity of local distributions for principle series which is derived from comparison the most continuous part of the relative trace formula (Offen).

If $(\lambda_1, \ldots, \lambda_n)$ are the parameters of π (or Π) the archimedean parts of the L-functions give

$$\sim \prod_{i<j} (\left|\lambda_i - \lambda_j\right| + 1)$$

If the finite part of the L-functions in the numerator and the denominator behave as it should (this is known for $n = 2$) then this quantity will roughly measure the size of the left-hand side.

In particular,

$$\|\varphi\|_\infty^2 \geq \prod_{i<j} (\left|\lambda_i - \lambda_j\right| + 1)^{1-\epsilon}$$

(cf. “trivial” upper bound $\prod_{i<j}(\left|\lambda_i - \lambda_j\right| + 1)^2$)

Another Example (work in progress): Q – quadratic form with integer coefficients of signature $(n,1)$. $\mathbb{H}^n = SO(n,1)/SO(n) = \{v \in \mathbb{R}^{n+1} : Q(v) = -1\}$ – n-dimensional hyperbolic space. $X = SO_{\mathbb{Z}}(Q)\backslash\mathbb{H}^n$.

$$\mathbb{H}^n \leftrightarrow \text{totally positive hyperplanes of } Q$$

Let Q' be the restriction of Q to a totally positive hyperplane defined over $\mathbb{Q}$ and let $\{x_i\}_{i=1}^{g} \subset X$ comprise the genus of Q'. If ϕ is a Maass form, which is a Hecke eigenfunction with $\Delta\phi + \lambda^2\varphi = 0$ then $\sum' \phi(x_i)$ (weighted by the size of the isometry groups of x_i) is zero unless ϕ is obtained as a theta lift of an automorphic form (of weight $\frac{1}{2}$ if n is even) ϕ' on an arithmetic quotient of $\mathbb{H}^2$ with eigenvalue $-\lambda^2$. Since the theta lifts constitute a rather meager set amongst the automorphic forms on X, the local Weyl law suggests that for these lifts $|\sum' \phi(x_i)|^2$ will be as large as λ^{n-2}, at least on average (compared to the trivial upper bound λ^{n-1}).

More refined analysis using “see-saw” formalism of Kudla.

$$\begin{tikzcd}
S\tilde{L}_2 \times \tilde{S}L_2 \arrow[dd, dash] & & SO(n,1) \arrow[dd, dash] \\
 & \times & \\
SL_2 & & SO(n) \times SO(1)
\end{tikzcd}$$

(for n odd). The formalism gives

$$\begin{aligned}
\sum{}'\phi(x_i) &= \int_{SO_{Q'}(\mathbb{Q})\backslash SO_{Q'}(\mathbb{A})} \phi(h)\ dh \\
&= \int_{SL_2(\mathbb{Q})\backslash SL_2(\mathbb{A})} \phi'(g)\theta_{SO'_Q}^{S\tilde{L}_2\times \tilde{S}L_2} 1(g)\ dg \\
&= \int_{SL_2(\mathbb{Q})\backslash SL_2(\mathbb{A})} \phi'(g)\tilde{\theta}(g)\tilde{E}(g, \frac{n}{2}-1)\ dg
\end{aligned}$$

a Shimura-type integral which is roughly $L((n-1)/2, \phi', Ad\otimes\chi_{Disc_Q})$. On the other hand, by the Rallis' inner product formula

$$\begin{aligned}
\|\varphi\|_2^2 &= L((n-1)/2, \phi', Ad\otimes\chi_{D_Q})\|\phi'\|_2^2 \\
&= L((n-1)/2, \phi', Ad\otimes\chi_{D_Q})L(1, \phi', Ad).
\end{aligned}$$

Thus,

$$\left(\frac{|\sum\phi(x_i)|}{\|\varphi\|_2}\right)^2 \sim \frac{L((n-1)/2, \phi', Ad\otimes\chi_{D_Q})}{L(1, \phi, Ad)}$$

Actually, the archimedean zeta integral is not unramified and this gives rise to correct behavior λ^{n-2}.

(It is possible to interpret the above formula for $\sum' \phi(x_i)$ in the framework of the relative formula. cf. Mao-Rallis)

More generally one would expect to detect other automorphic forms with large L^∞ norm by theta lifts from higher rank symplectic (or metaplectic) groups. The larger the rank is, the smaller the exponent will be in the lower bound on L^∞.

$$\begin{tikzcd} Sp_{2m}\times Sp_{2m} \arrow[d, dash] & & SO(n,1) \arrow[d, dash] \\ Sp_{2m} & >\!\!< & SO(n+1-m)\times SO(m-1,1) \end{tikzcd}$$

(m even, n odd)

$$\int_{SO(n+1-m)(\mathbb{Q})\backslash SO(n+1-m,\mathbb{A})} \theta\varphi(h)\ dh$$
$$= \int_{Sp_{2m}(\mathbb{Q})\backslash Sp_{2m}(\mathbb{A})} \varphi(h)E(h,(n-3m)/2)\theta(h)\ dh$$

where θ is the theta function of the pair $SO(m-1,1)\times Sp_{2m}$ (not integrated). This integral was considered by Adrianov ($m = 2$) and Piatetskii-Shapiro, Rallis (general case). It gives rise to $L((n-3m+1)/2,\pi,std)$ times a Fourier coefficient of the Siegel parabolic.

A similar phenomena will hold for locally symmetric spaces pertaining to $U(n,1)$.

General predictions by Sarnak:

1. For “most” automorphic forms $\|\varphi\|_\infty$ is small.
2. The exceptions are all accounted for by functoriality from smaller dimensional groups.
3. For the latter, $\|\varphi\|_\infty$ is controlled by behavior of φ on certain points or cycles, and its size is given in terms of the group from which φ originates.
4. The most optimistic conjecture:
$$acc\{\frac{\log\|\varphi\|_\infty}{\log\mu(\lambda)}\} \subset \{0, \frac{1}{2}, \dots, \frac{d-r-1}{2}\}.$$
where acc denotes accumulation points and φ range over automorphic forms on X

The analogy with the generalized Ramanujan's conjecture (as rephrased by Arthur) is evident. In fact, this is more in line with the original Ramanujan's conjecture which is about Fourier coefficients of automorphic forms, rather than temperedness of representations.

However, unlike the generalized Ramanujan's conjecture, there are exceptions already for GL_n as we have seen.

References

Hervé Jacquet: `www.math.columbia.edu/~hj`
A guide to the relative trace formula, conference proceedings, Columbus 2003

Erez Lapid: `www.math.huji.ac.il/~erezla`
The relative trace formula and its applications, conference proceedings, RIMS, Kyoto 2005

Omer Offen: `www.wisdom.weizmann.ac.il/~offen`

Peter Sarnak: `www.math.princeton.edu/~sarnak`
Letter to Cathleen Morawetz, 2004
Notes on the Generalized Ramanujan Conjectures, Fields Institute Lectures – June, 2003

Akshay Venkatesh: `www.math.nyu.edu/~venkatesh`