Residues of Eisenstein series

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Simplest Eisenstein series

$$E(z,s) = \sum_{\gamma \in \left(\begin{smallmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{smallmatrix}\right) \backslash SL_2(\mathbb{Z})} \mathrm{Im}(\gamma z)^{\frac{s+1}{2}}$$

$$= \frac{1}{\zeta(s+1)} \sum_{(m,n) \neq (0,0)} \frac{y^{\frac{s+1}{2}}}{|mz+n|^{s+1}}$$

Residue at $s = 1$ is

$$y \times \text{ density of lattice } \mathbb{Z} + \mathbb{Z}z \text{ in } \mathbb{C} \equiv 1$$

A parable: we have $\int_{SL_2(\mathbb{Z})\backslash\mathcal{H}} E(z,s)\, dz = 0$ for $-1 < \mathrm{Re}(s) < 1$. "Hence"

$$\int_{SL_2(\mathbb{Z})\backslash\mathcal{H}} \mathrm{res}_{s=1} E(z,s)\, dz = 0.$$

To overcome this "contradiction" define

$$\theta_f(z) = \sum_{\gamma \in \left(\begin{smallmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{smallmatrix}\right) \backslash SL_2(\mathbb{Z})} f(\mathrm{Im}\gamma z) \qquad f \in C_c^\infty(\mathbb{R}_+)$$

(finite sum, rapidly decreasing function). Let $\widehat{f}(s) = \int_{\mathbb{R}_+} f(x) x^{-s}\, d^*x$ - Mellin transform

$$\text{Mellin inversion:} \quad f(y) = \int_{\mathrm{Re}(s)=s_0} \widehat{f}(s) y^s\, ds$$

$$\theta_f(z) = \sum_{\left(\begin{smallmatrix}1 & \mathbb{Z} \\ 0 & 1\end{smallmatrix}\right)\backslash SL_2(\mathbb{Z})} \int_{\mathrm{Re}(s)=s_0} \widehat{f}(s)\mathrm{Im}(\gamma z)^s$$

$$= \int_{\mathrm{Re}(s)=s_0>>0} \widehat{f}(s)E(z, 2s-1)\ ds$$

$$= \widehat{f}(1)\mathrm{res}_{s=1}E(z,s) + \int_{\mathrm{Re}(s)=\frac{1}{2}} \widehat{f}(s)E(z,2s-1)\ ds$$

$$\int_{SL_2(\mathbb{Z})\backslash\mathcal{H}} \theta_f(z)\ dz = \int_{\gamma\in\left(\begin{smallmatrix}1 & \mathbb{Z} \\ 0 & 1\end{smallmatrix}\right)\backslash\mathcal{H}} f(\mathrm{Im}z)\ dz$$

$$= \int_0^\infty f(y)\frac{dy}{y^2} = \widehat{f}(1). \quad \text{OTOH}$$

$$\int_{SL_2(\mathbb{Z})\backslash\mathcal{H}} \theta_f(z)\ dz = \widehat{f}(1)\int_{SL_2(\mathbb{Z})\backslash\mathcal{H}} \mathrm{res}_{s=1}E(z,s)\ dz$$
$$+ \int_{\mathrm{Re}(s)=\frac{1}{2}} \widehat{f}(s)\int_{SL_2(\mathbb{Z})\backslash\mathcal{H}} E(z,2s-1)\ dz\ ds$$

We get $\int_{SL_2(\mathbb{Z})\backslash\mathcal{H}} E(z,s)\ dz = 0$ for $|\mathrm{Re}(s) < 1|$ and

$$\int_{SL_2(\mathbb{Z})\backslash\mathcal{H}} \mathrm{res}_{s=\frac{1}{2}}E(z,s)\ dz = 1$$

OTOH, the constant term of $\mathrm{res}_{s=\frac{1}{2}}E(z,s)$ is $\mathrm{res}_{s=1}\zeta(s)/\zeta(2)$ so we obtain $\mathrm{vol}(SL_2(\mathbb{Z})\backslash\mathcal{H}) = \zeta(2)$.

In general, if G is semi-simple and split over F, $B = TU$ Borel, $\mathfrak{a}_0^* = X^*(T) \otimes \mathbb{R}$, $\mathfrak{a}_0 = X_*(T) \otimes \mathbb{R}$,

$$H_0(tuk) = H_0(t) \quad e^{\langle \chi, H_0(t) \rangle} = |\chi(t)| = \prod_v |\chi(t_v)|_v$$

Define Eisenstein series and pseudo Eisenstein series by

$$E(g, \lambda) = \sum_{B(F)\backslash G(F)} e^{\langle \lambda + \rho, H_0(g) \rangle}$$

$$\theta_\phi(g) = \sum_{B\backslash G} f(H_0(\gamma g)) = \int_{\mathrm{Re}\lambda = \lambda_0} \hat{f}(\lambda) E(g, \lambda)\ d\lambda$$

for $f \in C_c^\infty(\mathfrak{a}_0)$, $\lambda_0 >> 0$ and

$$\hat{f}(\lambda) = \int_{\mathfrak{a}_0} f(X) e^{-\langle \lambda + \rho, X \rangle}\ dX$$

$$\int_{G(F)\backslash G(\mathbb{A})} \theta_\phi(g) = \int_{B(F)\backslash G(\mathbb{A})} f(H_0(g))\ dg$$

$$= \int_K \int_{U(F)\backslash U(\mathbb{A})} \int_{T(F)\backslash T(\mathbb{A})} f(H_0(tuk))\ dt\ du\ dk =$$

$$\mathrm{vol}(T(F)\backslash T(\mathbb{A})^1) \int_{\mathfrak{a}_0} e^{-\langle 2\rho, X \rangle} f(X)\ dX = \mathrm{vol}(T) \hat{f}(\rho)$$

$$\Longrightarrow \int_{G(F)\backslash G(\mathbb{A})} \mathrm{res}_{\lambda = \rho} E(g, \lambda)\ dg = \mathrm{vol}(T)$$

On the other hand since $\text{res}_{s=\lambda} E(g,\lambda)$ is constant, it is given by its constant term.

$$\text{res}_{s=\lambda} E(g,\lambda) = \lim_{\lambda \to \rho} \prod_{\alpha \in \Delta_0} (\left\langle \lambda, \alpha^\vee \right\rangle - 1) m(\lambda)$$

By a well-known computation

$$m(\lambda) = \prod_{\alpha \in \Phi_+} \frac{\zeta_F(\langle \lambda, \alpha^\vee \rangle)}{\zeta_F(\langle \lambda, \alpha^\vee \rangle + 1)}$$

$$\Longrightarrow \text{res}_{s=\lambda} E(g,\lambda) = \prod_{\alpha \in \Delta_0} \zeta_{-1} \prod_{i>1} \zeta_F(i)^{n_i}$$

$$\begin{aligned} \text{where } n_i =& \#\{\alpha \in \Phi_+ : \left\langle \lambda, \alpha^\vee \right\rangle = i\} \\ & -\#\{\alpha \in \Phi_+ : \left\langle \lambda, \alpha^\vee \right\rangle = i-1\} \end{aligned}$$

and $\zeta_{-1} = \text{res}_{s=1} \zeta_F(s) = \text{vol}(F^* \backslash \mathbb{I}_F^1)$. For example, for $SL(n)$, $n_i = 1$, $i = 2, \ldots, n$. Also

$$\text{vol}(T(F) \backslash T(\mathbb{A})^1) = \zeta_{-1}^{\text{rank}(G)}$$

so finally we get

$$\boxed{\text{vol}(G(F) \backslash G(\mathbb{A})) = \prod_{i>1} \zeta_F(i)^{n_i}} \quad (\textbf{Langlands})$$

Degenerate Eisenstein series on $GL(n)$, $Sp(n)$

Let GL_n acts on $\mathbb{V}$ from the right. For a Schwartz-Bruhat function Φ on $\mathbb{V}(\mathbb{A}_F)$

$$\mathcal{E}(g,s) = \int_{F^*\backslash \mathbb{I}_F} \sum_{v\in\mathbb{V}-\{0\}} \Phi(vtg)|t|^{m(s+1)/2}\ d^*t$$

This is the Eisenstein series induced from the mirabolic subgroup. Analytic continuation (and properties) are obtained by Tate's thesis. Set

$$\Theta_\Phi(g) = \sum_{v\in\mathbb{V}} \Phi(vg) \quad \Theta^*_\Phi(g) = \Theta_\Phi(g) - \Phi(0)$$

Poisson summation: $\Theta_\Phi(g) = |\det g|^{-1}\Theta_{\widehat{\Phi}}(g^*)$

where $\widehat{\Phi}(x) = \int_{\mathbb{V}(\mathbb{A})} \Phi(y)\psi_0([x,y])\ dy.$

$$\mathcal{E}_\Phi(g,s) = \int_{F^*\backslash \mathbb{I}_F} \Theta^*_\Phi(tg)|t|^{n(s+1)}\ d^*t$$

$$= \int_1^\infty \int_{F^*\backslash \mathbb{I}^1_F} \Theta^*_\Phi(txg)\ d^*x\ t^{n(s+1)}\ d^*t$$

$$+ \int_1^\infty \int_{F^*\backslash \mathbb{I}^1_F} \Theta^*_{\widehat{\Phi}}(txg^*)\ d^*x\ t^{n(1-s)}\ d^*t$$

$$-\zeta_{-1}\cdot\Phi(0)/(n(s+1)) + \zeta_{-1}\cdot\widehat{\Phi}(0)/(n(s-1))$$

$$= \mathcal{E}_{\widehat{\Phi}}(g^*,-s)$$

The restriction of $\mathcal{E}(g,s)$ to $H = Sp(2n)$ gives the Eisenstein series of H, since $v_0 H = \mathbb{V} - \{0\}$.
Pseudo-Eisenstein series: for $f \in C_c^\infty(\mathbb{R}_+)$

$$\theta_{\Phi,\sigma}(g) = \int_{F^*\backslash \mathbb{I}_F} \sum_{\mathbb{V}-\{0\}} \Phi(vtg) f(t)\ d^*t =$$

$$= \int_{\mathrm{Re}(s)=s_0>>0} \hat{f}(s) \mathcal{E}_\Phi(h,s)\ ds$$

Cuspidal Eisenstein series in co-rank 1
G reductive, $P = MU$ maximal parabolic, α - simple root for $Z(M)$ on U, $\alpha^\vee$ - the co-root, $\varpi = \frac{\alpha}{\langle \alpha, \alpha^\vee \rangle}$ fundamental weight, φ a cuspidal automorphic form on $A_M M(F) U(\mathbb{A}) \backslash G(\mathbb{A})$, $A_M = \{\alpha^\vee(t) : t \in \mathbb{R}_+\}$. Eisenstein series

$$\sum_{P(F)\backslash G(F)} \varphi(\gamma g) e^{\langle s\varpi + \rho_P, H_0(\gamma g) \rangle}$$

intertwining operator (w.r.t w-longest Weyl element)

$$M(s)\varphi(g) = \int_{U(\mathbb{A})} \varphi(wug) e^{\langle s\varpi + \rho_P, H_0(wug) \rangle}\ du$$

Basic properties:

• absolutely convergent for $\mathrm{Re}(s) > \rho_P$

• meromorphic continuation and functional equations $\quad E(g, \varphi, s) = E(g, M(s)\varphi, -s)$

$$M(-s)M(s) = I \quad M(s)^* = M(\bar{s})$$

• $E(g, \varphi, s)$, $M(s)$ are holomorphic for $\mathrm{Re}(s) = 0$.

• $E_U = \varphi e^{\langle s\varpi + \rho_P, H_0(g)\rangle} + M(s)\varphi e^{\langle -s\varpi + \rho_P, H_0(g)\rangle}$ if $wMw^{-1} = M$

• The singularities of $E(g, \varphi, s)$ and $M(s)\varphi$ are the same. There are finitely many of them for $\mathrm{Re}(s) > 0$. They are real and simple, and can occur only if $wMw^{-1} = M$ and $w\pi \simeq \pi$. If s_0 is a pole and $E_{-1}(g, \varphi) = \mathrm{res}_{s=s_0} E(g, \varphi, s)$, $M_{-1}\varphi = \mathrm{res}_{s=s_0} M(s, \varphi)$ then E_{-1} is square-integrable and

$$(E_{-1}\varphi, E_{-1}\varphi)_{G(F)\backslash G(\mathbb{A})^1} = (M_{-1}\varphi, \varphi)_{M(F)\backslash M(\mathbb{A})^1 \times K}$$

• If G is quasi-split and π is generic, poles only (possibly) for $s = \frac{1}{2}$ or $s = 1$.

Basic questions: what are the properties on π for poles to exist. What can we said about residual representations?

First possible answer - poles of L-functions. Another possibility - non-vanishing of periods. These answers may be related by an integral presentation of the L-function. The "inner period" may reflect an "outer period" of the residue representation. The inducing representation and the residual representation are related by functoriality, and hence, possibly by a trace identity.

Example $G = GL_{2n}$, $P = MU$ parabolic of type (n, n),

$$E(g, \varphi, s) = \sum_{\gamma \in P \backslash G} \varphi(\gamma g) e^{\langle s\varpi + \rho_P, H_0(\gamma g)\rangle}$$

Eisenstein series on $G(\mathbb{A})$ induced from a cuspidal section φ on $A_M M(F) U(\mathbb{A}) \backslash G(\mathbb{A})$ where

$A_M = \{\begin{pmatrix} tI_n & 0 \\ 0 & t^{-1}I_n \end{pmatrix} : t \in \mathbb{R}_+\}$. Let $E_{-1}(\cdot, \varphi)$ be the residue of $E(\cdot, \varphi, s)$ at $s = 1$.

Jacquet-Rallis: $\int_{H(F)\backslash H(\mathbb{A})} E_{-1}(h, \varphi)\ dh$

$$= \int_{K_{Sp_{2n}}} \int_{GL_n(F)\backslash GL_n(\mathbb{A})^1} \varphi(\begin{pmatrix} g & 0 \\ 0 & {}^t g^{-1} \end{pmatrix} k)\ dg\ dk$$

This implies that $E_{-1} \neq 0$ for $\pi \otimes \pi$.

Proof Introduce pseudo-Eisenstein series

$$\theta(g) = \sum_{\gamma \in P\backslash G} F(\gamma g)$$

$$= \int_{\mathrm{Re}(s) = s_0 >> 0} \widehat{F}(s)(g) E(s, \varphi, g)\ ds$$

This is rapidly decreasing where $F(\cdot)$ - cuspidal function on $M(F)U(\mathbb{A})\backslash G(\mathbb{A})$ s.t. $\forall g \in G(\mathbb{A})$, $F(\cdot g)$ is compactly supported in A_M.

$$\widehat{F}(s)(g) = \int_{A_M} F(tg) e^{-\langle \rho_P + s\varpi, H_0(t) \rangle}\ dt$$

$$\int_{H(F)\backslash H(\mathbb{A})} \theta(h)\ dh = \sum_{\eta \in P\backslash G/H} \int_{H_\eta \backslash H(\mathbb{A})} F(\eta h)\ dh$$

$$P\backslash G/H \leftrightarrow \{0 \le r \le n,\ r \text{ even}\}$$

$$PgH \mapsto \text{rank}([\cdot,\cdot]|_{V_0 g})$$

$$V_0 = (\overbrace{0,\ldots,0}^{n},\overbrace{*,\ldots,*}^{n}) \quad (\text{Stab}(V_0) = P)$$

Contribution from $0 < r < n$ is zero by cuspidality. Contribution from $r = n$ factors through

$$\int_{Sp_n(F)\backslash Sp_n(\mathbb{A})} \varphi(h)\ dh = 0$$

Contribution from $r = 0$ is $\int_{P_H \backslash H(\mathbb{A})} F(h)\ dh =$

$$\int_{\substack{k \in K_{Sp_{2n}} \\ g \in GL_n(F)\backslash GL_n(\mathbb{A}) \\ t \in A_M}} \delta_{P_H}(t)^{-1} F(t \begin{pmatrix} g & 0 \\ 0 & {}^t g^{-1} \end{pmatrix} k)\ dt\ dg\ dk$$

$$= \int_{K_{Sp_{2n}}} \int_{GL_n(F)\backslash GL_n(\mathbb{A})} \widehat{F}(1)(\begin{pmatrix} g & 0 \\ 0 & {}^t g^{-1} \end{pmatrix} k)\ dt\ dg\ dk$$

since $2\rho_{P_H} = \rho_P + \varpi$.

Miraculously there exists an alternative formula for the same expression (**Jacquet-L-Rallis**)

$$\int_{H\backslash H(\mathbb{A})} E_{-1}(h,\varphi)\ dh$$

$$= \int_{K_H} \int_{GL_n(F)\backslash GL_(\mathbb{A})^1} M_{-1}\varphi(\begin{pmatrix} g & 0 \\ 0 & g^* \end{pmatrix} k)\mathcal{E}(g,3)\ dg\ dk$$

$$(= \int_{K_H} \int_{GL_n(F)\backslash GL_(\mathbb{A})^1} \varphi(\begin{pmatrix} g & 0 \\ 0 & g^* \end{pmatrix} k)\ dg\ dk)$$

Why are these two expressions equal?
On the one hand $M_{-1} = \frac{\mathrm{res}_{s=1} L(s,\pi\otimes\widetilde{\pi})}{L(2,\pi\otimes\widetilde{\pi})}$. On the other hand we have the Jacquet-Shalika integral

$$\int_{GL_n(F)\backslash GL_n(\mathbb{A})} \varphi(\begin{pmatrix} g & 0 \\ 0 & g^* \end{pmatrix})\mathcal{E}(g,2s-1)\ dg = L(s,\pi\otimes\widetilde{\pi})$$

Bessel distributions and weighted traces

Definition: Let (π, V) be an admissible representation, $(\pi^\vee, V^\vee)$ its admissible dual. For any forms l on V and l' on $V^\vee$ define

$$\mathcal{B}^{\pi}_{l,l'}(f) = l'\left[l \circ \pi(f)\right]$$

for $f \in C_c^\infty(G)$. This is the Bessel distribution of π with respect to l, l'.

Similarly, if $\mathcal{L}$ is a bilinear form on $V^\vee \times V$ the weighted trace is define by

$$\mathcal{T}^{\pi}_{\mathcal{L}}(f) = \mathcal{L}(\pi(f))$$

for $f \in C_c^\infty(G)$ where $\pi(f) \in \mathrm{End}(V) \simeq V^\vee \otimes V$. (If $\mathcal{L}$ is standard pairing this gives the usual trace.) The relation is

$$\mathcal{B}^{\pi}_{l,l'}(f) = \mathcal{T}^{\pi}_{l' \otimes l}(f).$$

$$\mathcal{B}^{\pi \otimes \pi^\vee}_{l,\mathcal{L}}(f_1 \otimes f_2) = \mathcal{T}^{\pi}_{\mathcal{L}}(f_2^\vee \star f_1)$$

where $f_2^\vee(g) = f_2(g^{-1})$.

If π is unitary, $\overline{V} \simeq V^\vee$. For forms l_1, l_2 on V we have

$$\mathcal{B}_{l_1,\overline{l_2}} = \sum_{\{\varphi\}} l_1(\pi(f)\varphi) l_2(\varphi) \quad \{\varphi\} \text{ - O.N. basis}$$

Back to our case, π - cuspidal representation of $GL_n(\mathbb{A})$, Π residual representation on $GL_{2n}(\mathbb{A})$ - quotient of $\mathrm{Ind}(\pi|\det\cdot|^{\frac{1}{2}} \otimes \pi|\det\cdot|^{-\frac{1}{2}})$. Set

$$\ell_H(\varphi) = \int_{H(F)\backslash H(\mathbb{A})} \varphi(h)\ dh \quad \varphi \in V_\Pi \quad \phi, \phi' \in V_\pi$$

$$\mathfrak{B}(\phi, \phi') = \int_{GL_n(F)\backslash GL_n(\mathbb{A})^1} \overline{\phi(x)}\phi'(x)\mathfrak{E}(x^*, 3)\ dx$$

Theorem (Jacquet-L-Rallis)

$$\boxed{\mathcal{B}^{\Pi}_{\ell_H, \bar{\ell}_H}(f) = \mathcal{T}^{\pi}_{\mathfrak{B}}(f')}$$

$$\boxed{\mathcal{B}^{\Pi}_{\ell_H, \overline{\mathcal{W}^{\psi}}}(f) = \mathcal{B}^{\pi}_{\mathcal{W}^{\psi'}, \overline{\mathcal{W}^{\psi'}}}(f')}$$

where $f'(g') =$

$$\int_{\substack{k, k' \in \mathbf{K}_H \\ u \in U(\mathbb{A}) \\ x \in GL_n(\mathbb{A})}} |\det(x)|^{n+1} \cdot f(k' \begin{pmatrix} xg' & 0 \\ 0 & x^* \end{pmatrix} uk).$$

(basically Harish-Chandra map) and $\mathcal{W}^{\psi}$ is Fourier coefficient w.r.t $\psi(\begin{pmatrix} n_1 & * \\ 0 & n_2 \end{pmatrix}) = \psi'(n_1)\psi'(n_2)$ where $\psi'((x_{i,j})) = \theta(x_{12} + \cdots + x_{n-1,n})$

The theorem gives spectral identities between the spectral sides of two pairs of trace formulas. However, the result is proved without appealing to the trace formula!. (The geometric side of the second trace formula had in fact been worked out by **Jacquet-Rallis**).

The second spectral identity is a formal consequences of the following

- $\int_{H(F)\backslash H(\mathbb{A})} E_{-1}(h, \varphi)\ dh = \int \varphi(mk)\ dm\ dk$
- $(E_{-1}\varphi, E_{-1}\varphi) = (M_{-1}\varphi, \varphi)$
- $\mathcal{W}^{\psi}(E_{-1}\varphi) = \mathcal{W}^{\psi'}(M_{-1}\varphi)$

For the first spectral identity we need the other formula for $\int_{H(F)\backslash H(\mathbb{A})} E_{-1}(h, \varphi)\ dh$ in terms of $M_{-1}\varphi$.

Essential for the above to work is an explicit construction of the functoriality (as well as an expression for the L^2-norm of this construction). There are other cases where the functoriality is given explicitly (more complicated residual Eisenstein series, theta-correspondence, descent construction,...) The hope is to use the same method to prove other trace identities.

Proof of second formula for $\int_{H(F)\backslash H(\mathbb{A})} E_{-1}(h,\varphi)$ Use pseudo-Eisenstein series. More precisely,

$$\int_{H\backslash H(\mathbb{A})} E_{-1}(h,\varphi)\theta_{\Phi,f}(h)\ dh = \hat{f}(1)\times$$
$$\int_{K_H}\int_{M_H\backslash M_H(\mathbb{A})^1} M_{-1}\varphi(mk)\ \mathbb{E}_{\Phi}(mk,3)\ dm\ dk.$$

This implies the result and also that

$$\int_{H\backslash H(\mathbb{A})} E_{-1}(h,\varphi)\mathcal{E}_{\Phi}(h,s)\ dh = 0 \quad |\mathrm{Re}(s)| < 1.$$

Recall that for φ cuspidal on GL_n we have (**Piatetski-Shapiro**)

$$\varphi(g) = \sum_{\gamma\in Z_n(F)U_0(F)\backslash P_{n-1,1}(F)} \mathcal{W}^{\psi}_{\varphi}(\gamma g)$$

For E_{-1} we have (**Moeglin-Waldspurger**)

$$E_{-1}(g) = \sum_{\gamma \in P_{n,n-1,1} \backslash P_{2n-1,1}} M_{-1}\varphi(\gamma g)$$

$$\int_{Q_H \backslash H(\mathbb{A})} E_{-1}(h,\varphi) \int_{\mathbb{I}_F} \Phi(v_0 th) f(t) d^*t \; dh =$$

$$\sum_{\eta \in P_{(n,n-1,1)} \backslash P_{(2n-1,1)}/Q} \int_{Q_\eta \backslash H(\mathbb{A})} M_{-1}\varphi(\eta h)$$

$$\int_{\mathbb{I}_F} \Phi(v_0 th) f(t) d^*t \; dh$$

$$P_{(n,n-1,1)} \backslash P_{(2n-1,1)} \leftrightarrow$$

n-dim subspaces $\mathbb{V}_0$ of $\mathbb{V}$ containing v_0

The Q-orbits are parameterized by $r = \text{rank}([\bullet,\bullet]|_{\mathbb{V}_0})$ and whether or not $v_0 \in \text{rad}([\bullet,\bullet]|_{\mathbb{V}_0})$.

$r = 0$ (closed orbit) gives desired contribution; $r = n$ vanishes because symplectic period for cusp form is 0; intermediate orbits vanish by cuspidality.

There are other cases where we have

$$\int_{H(F)\backslash H(\mathbb{A})} E_{-1}(h,\varphi)\ dh$$
$$= \int_{K_H} \int_{M_H\backslash M_H(\mathbb{A})\cap M(A)^1} \varphi(mk)\ dm\ dk$$

for a period subgroup H, $M_H = M \cap H$.
Example: (**Ginzburg-Rallis-Soudry**) $G = Sp(4n)$, P - Siegel parabolic, $M = GL(2n)$, $H = Sp(2n) \times Sp(2n)$, $M_H = GL(n) \times GL(n)$.

$$\int_{M_H(F)\backslash M_H(\mathbb{A})\cap M(\mathbb{A})^1} \phi((g_1,g_2))\mathcal{E}(g_1,s_1)$$
$$|\frac{\det g_1}{\det g_2}|^{s_2-\frac{1}{2}}\ dg_1\ dg_2 = L(s_1,\pi,\wedge^2)L(s_2,\pi)$$

(**Bump-Friedberg**) M_H-period $\neq 0 \iff$ for $s_2 = \frac{1}{2}$ $\exists$ pole at $s_1 = 1$, i.e. $L(s,\pi,\wedge^2)$ has a pole at $s = 1$ and $L(\frac{1}{2},\pi) \neq 0$. This is exactly the condition for a pole for the Eisenstein series at $s = \frac{1}{2}$.
We get a spectral identity

$$\boxed{\mathcal{B}^{\Pi}_{\ell_H,\overline{\mathcal{W}^{\psi}}}(f) = \mathcal{B}^{\pi}_{\ell_{M_H},\overline{\mathcal{W}^{\psi'}}}(f')} \qquad \ell_{M_H} = M_H\text{-period}$$

where $\psi(\begin{pmatrix} u & * \\ 0 & u^* \end{pmatrix}) = \psi'(u)$, ψ' is a generic character for GL_{2n}. In order to obtain a spectral identity $\boxed{\mathcal{B}^{\Pi}_{\ell_H,\bar{\ell}_H}(f) = \mathcal{B}^{\pi}_{\ell_{M_H},\bar{\ell}_{M_H,\mathfrak{E}}}(f')}$ where $\ell_{M_h,\mathfrak{E}}$ is a "co-period" functional we need a formula

$$\int_{H(F)\backslash H(\mathbb{A})} E_{-1}(h,\varphi)\ dh$$
$$= \int_{K_H} \int_{M_H \backslash M_H(\mathbb{A}) \cap M(\mathbb{A})^1} M_{-1}\varphi\mathcal{E}(m,3)\ dm\ dk$$

The main issue is to have a formula of E_{-1} in terms of M_{-1}, analogous to Mœglin-Waldspurger. We expand E_{-1} along the center Y_{2n} of unipotent radical of V_{2n} of the parabolic whose Levi is $GL(1) \times Sp(4n-2)$. The constant term can be further expanded over V_{2n}. The zeroth coefficient vanishes by cuspidality. The nonzero coefficients are one orbit under the Levi. We continue this way. When we get to the non-degenerate Fourier coefficient in GL_{2n}, we can use Piatetski-Shapiro's formula. However, there are many accompanying terms.

More precisely, let N_k' be the inverse image in $P_{k;2n-k}$ of the maximal unipotent of GL_k under the projection map to the Levi factor. Let $N_k = N_{k-1}' Y_{2n-k+1}$ (adding the one-dimensional center of a smaller Heisenberg unipotent radical). Define $\psi_k^\alpha(x) = \theta(x_{1,2} + x_{2,3} + \cdots + x_{k-1,k} + \alpha x_{k,2n-k})$. Let $f = E_{-1}(\cdot, \varphi)$. Then

$$f(g) = \sum_{(Q \cap P)\backslash Q} M_{-1}\varphi(\gamma g) + \sum_{k=1}^{2n} f_k(g) \quad \text{where}$$

$$f_k(g) = \sum_{\gamma \in Z_k N_k' Sp_{4n-2k} \backslash Q} \sum_{\alpha \in F^*} \mathcal{W}_f^{N_k, \psi_k^\alpha}(\gamma g)$$

We can show that the terms f_k do not contribute to the H-period. The analysis of the terms coming from $M_{-1}\varphi$ is similar to before and gives the required result. Problem - very difficult convergence issues.

Ginzburg-Rallis-Soudry showed that $f_k \equiv 0$ for $k > n$ and f_n gives the descent from π to $\widetilde{Sp_n}$. This is analogous to **Bernstein-Zelevinski**'s construction of derivatives.

Goal: obtain a spectral identity for descent construction. **Mao-Rallis** worked out a series of trace formulas comparisons, leading eventually to descent. The above is the spectral counterpart of the first in their series...

Jiang gave a setup for evaluating outer periods of residual Eisenstein series in terms of inner periods. In particular, he relates G_2-periods on $SO(8)$ to triple products, and applies this to Jacquet's conjecture.

Residual Eisenstein series and positivity

A cuspidal representation π of $GL_n(\mathbb{A})$ is self-dual if and only if $L(s, \pi \otimes \pi)$ has a pole at $s = 1$.

$$L(s, \pi \otimes \pi) = L(s, \pi, \mathrm{sym}^2) L(s, \pi, \wedge^2)$$

Self-dual cuspidal representations are classified to *symplectic* and *orthogonal* according to which L-function has a pole.

This is similar to the classification of self-dual representations of finite (or compact) groups (and suggested by the Tannakian formalism of Langlands).

In particular, if π is symplectic then n is even and the central character is trivial.

Working with residual Eisenstein series on classical groups we get

Theorem (**L.–Rallis**) If π is symplectic then $L(\frac{1}{2}, \pi) \geq 0$.

Theorem If π is symplectic and π' is orthogonal then $L(\frac{1}{2}, \pi \otimes \pi') \geq 0$.

Remarks:

- This would be a trivial consequence of GRH. $L(\frac{1}{2}, \pi)$ appears in BSD, theta-correspondence, Gross-Prasad and Jacquet's conjectures. If π is motivic and the center of symmetry is a critical point in the sense of Deligne then π is symplectic.

• For $n = 2$, π is symplectic if and only if $\omega_\pi = 1$. In this case non-negativity follows from an explicit formula. (**Waldspurger**, **Zagier**, **Katok–Sarnak** – using **Shimura** correspondence, **Jacquet**, **Guo**, **Baruch-Mao** – relative trace formula, **Sarnak** - quantum chaos). It already has striking applications to upper bounds of $L(\frac{1}{2}, \pi)$ for $\pi=$ quadratic Dirichlet character (**Conrey-Iwaniec**), or Maass form (**Ivic**).

• For the standard L-function we can take the partial L-function. For the tensor product it is necessary to take the *completed* L-function as defined by **Jacquet–Piatetski-Shapiro–Shalika**; the local factors are not known to be positive by lack of **Ramanujan** hypothesis. (We need a bound of $\frac{1}{4}$ on the non-tempered parameters. Best known bound is $\frac{1}{2} - \frac{1}{n^2+1}$ (**Luo-Rudnick-Sarnak**).

• Using the tensor product functoriality from $GL_2 \times GL_2$ to GL_4 (Ramakrishnan) we obtain as a corollary that $L(\frac{1}{2}, \pi_1 \otimes \pi_2 \otimes \pi_3) \geq 0$ if π_i are on PGL_2. (Compatible with **Harris–Kudla**, **Böcherer–Schultze-Pillot**, **Watson**)

• For the tensor product we use the functoriality $SO(2n+1) \leftrightarrow GL_{2n}$ proved by **Ginzburg–Rallis–Soudry**, **Cogdell–Kim–Piatetski-Shapiro-Shahidi**, **Jiang–Soudry**: There exists a (functorial) one-to-one correspondence between generic cuspidal representations of $SO(2n+1)$ and families $(\tau_1, \dots, \tau_k)$ of distinct cuspidal symplectic representations of GL_{n_i} with $\sum n_i = 2n$

Epsilon factors
Let $G_{2n+1} = Sp(2n)$, $G_{2n} = SO(2n)$ (split).
Theorem Let π be a generic representation of G_n over a local field of characteristic 0.

$$\text{Then} \quad \boxed{\epsilon(\frac{1}{2}, \pi) = \pi(-1)}.$$

If Π is a generic automorphic cuspidal representation of $G_n(\mathbb{A})$ then $\epsilon(\frac{1}{2}, \pi) = 1$.

Remarks

• Motivation: if $\varphi : W_F \to SO(n, \mathbb{C})$ $\epsilon(\frac{1}{2}, \varphi)$ detects whether φ lifts to $Spin(n, \mathbb{C})$ (**Deligne**). Under local Langlands reciprocity this condition translates to whether π comes from $PSp(n-1, F)$ (n odd) or $PSO(n, F)$ (n odd).

Note: $PSp(2n, F) \neq Sp(2n, F)/\{\pm 1\}$, but rather $PSp(2n, F) = PGSp(2n, F)$.

The global result is an analogue of **Frölich-Queryut** (Galois groups), **Saito** (motives).

• For quasi-split special even orthogonal groups of discriminant η we expect

$$\boxed{\epsilon(\frac{1}{2}, \pi, \psi) = \pi(-1) \cdot \epsilon(\frac{1}{2}, \eta, \psi)}$$

• Orthogonal cuspidal representations of GL_n (with trivial central character) descend to G_n (**GRS**). Once the expected properties of the descent are established, it would follow that (globally) $\epsilon(\frac{1}{2}, \pi) = 1$. For $n = 1$

$$\sum_{j=1}^{p-1} e^{2\pi i j/p} \left(\frac{j}{p}\right) = +\sqrt{(-1)^{(p-1)/2} p}. \quad \textbf{(Gauss)}$$

• Hopefully, it will be possible to eliminate the genericity (as well as split) assumption using the "doubling method" of **Rallis–Piatetski-Shapiro**.

• Using **GRS-CKPSS-JS** can prove: If π, π' are generic representations of $SO(2n+1)$, $SO(2m+1)$ over a local field then $\epsilon(\frac{1}{2}, \pi \otimes \pi') = 1$. (special cases - **D. Prasad-Ramakrishnan**)

• Granted the properties of the other lifts from classical groups: if π, π' are generic representations of G_n, G_m (locally) then $\epsilon(\frac{1}{2}, \pi \otimes \pi') = \pi(-1)^m \pi'(-1)^n$. Globally it is 1.

Proof $n = 2$ case. $SO(2) \simeq GL(1)$.

$$\epsilon(\frac{1}{2}, \chi, std, \psi) = \epsilon(\frac{1}{2}, \chi, \psi)\epsilon(\frac{1}{2}, \chi^{-1}, \psi)$$
$$= \epsilon(\frac{1}{2}, \chi, \psi)\chi(-1)\epsilon(\frac{1}{2}, \chi^{-1}, \bar{\psi}) = \chi(-1)$$

$F = \mathbb{C}$ follows (every generic is principal series). p-adic case: Reduction to supercuspidal case - routine. Let k be totally complex, $k_{v_0} = F$, Π cuspidal generic, unramified outside archimedean places and v_0, $\Pi_{v_0} = \pi$.

$$\epsilon(\frac{1}{2}, \Pi) = \epsilon(\frac{1}{2}, \pi) \cdot \prod_{v \text{ complex}} \epsilon(\frac{1}{2}, \Pi_v)$$

Global result in this case implies local result.
theta correspondence from G_n to G_{n+1}
Theorem [**Mœglin, Ginzburg-Rallis-Soudry**]
Π - generic cuspidal representation of $G_n(\mathbb{A})$. Either Π is the theta-lift of a cuspidal generic representation of $G_{n-1}(\mathbb{A})$ (iff $L^S(1, \Pi) = \infty$), or the theta-lift Π' to G_{n+1} is cuspidal generic irreducible, in which case, $L^S(s, \Pi') = L^S(s, \Pi)\zeta_F^S(s)$ has a pole at $s = 1$.

Can assume we are in the first case. Let $E(g, \varphi, s)$ be the Eisenstein series from $(GL(1) \times G_n, \mathbf{1} \otimes \Pi)$ to G_{n+2}. It has a residue at $s = 1$.

$$\boxed{(E_{-1}\varphi, E_{-1}\varphi)_{\underline{G_{n+2}}} = (M_{-1}\varphi, \varphi)_{\underline{M} \times K}}$$

Normalize the local intertwining operators (Shahidi)

$$M(\Pi_v, s) = \frac{L_v(s, \Pi_v)}{\epsilon_v(s, \Pi_v, \psi_v) L_v(s+1, \Pi_v)} R_v(\Pi_v, s, \psi_v)$$

$$M_{-1} = \frac{\mathrm{res}_{s=1} L(s, \Pi)}{\epsilon(1, \Pi) L(2, \Pi)} \otimes R_v(\Pi_v, s, \psi_v)$$

M_{-1} is positive definite + $L(s, \Pi)$ has no zeros or poles for $\mathrm{Re}(s) > 1 \implies R(\Pi, 1)$ is semi-definite of the same sign as $\epsilon(\frac{1}{2}, \Pi)$.
Show that $R_v(\Pi_v, 1, \psi_v)$ is positive $\forall v$.
Unramified places - obvious.
Complex places - multiplicativity.
For v_0, $I(\pi, s)$ is irreducible for $0 \leq s < 1$. $R(\pi, s, \psi)$ is a family of continuous Hermitian operators, non-degenerate for $0 \leq s < 1$. Also, $R(\pi, 0, \psi)$ has a non-trivial $+1$-eigenspace (the ψ-generic part of $I(\pi, 0)$) (Keys-Shahidi).

Remark: For any irreducible representation π of GL_n over a local field

$$\epsilon(\frac{1}{2}, \pi \otimes \widetilde{\pi}) = \omega_\pi(-1)^{n-1}$$

(**Bushnell–Henniart**).

In the square-integrable case, here's an argument of **Jacquet** using the **Jacquet–Piatetski-Shapiro–Shalika** zeta integral $\Psi(s, W, W', \Phi)$ for the **Rankin-Selberg** L-function

$$\int_{N_n \backslash GL_n} W(g) W'(g) \Phi((0, \ldots, 0, 1)g) |\det g|^s \, dg$$

which converges for $\mathrm{Re}(s) > 0$. Here W, W' are elements in the Whittaker spaces of π and $\widetilde{\pi}$ respectively, and Φ is a Schwartz function on F^n. The local functional equation is

$$\Psi(\frac{1}{2}, \widetilde{W}, \overline{\widetilde{W}}, \hat{\Phi}) \\ = \epsilon(\frac{1}{2}, \pi \otimes \widetilde{\pi}, \psi) \omega_\pi(-1)^{n-1} \Psi(\frac{1}{2}, W, \overline{W}, \Phi)$$

For an appropriate Φ, the zeta-integrals on either side are positive!