

# A REMARK ON EISENSTEIN SERIES

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## 1. INTRODUCTION

The theory of Eisenstein series is fundamental for the spectral theory of automorphic forms. It was first developed by Selberg, and was completed by Langlands ([Lan76]; see also [MW95]). There are several known proofs for the meromorphic continuation of Eisenstein series (apart from very special cases of Eisenstein series which can be expressed in terms of Tate integrals). In all these proofs it is convenient, if not essential, to assume (in the number field case) that the inducing section is  $K$ -finite, to ensure finite dimensionality. However, the analytic properties of Eisenstein series are closely tied to, and at any rate controlled by, those of the intertwining operators. The latter decompose into local intertwining operators. In the archimedean case, a lot is known about the local intertwining operators and no  $K$ -finiteness assumption on the section is necessary. It is therefore reasonable to expect that the analytic properties of Eisenstein series for a general smooth section follow from that of  $K$ -finite sections. The modest goal of this short note is to carry this out (using the meromorphic continuation of  $K$ -finite Eisenstein series as a black box). In fact, by the automatic continuity theorem of Casselman and Wallach ([Wal92, Ch. 11]), at each regular point the Eisenstein series can be extended to smooth sections. This by itself does not suffice to prove meromorphic continuation unless one knows some local uniformity (in the spectral parameter) for the modulus of continuity of the Eisenstein series as a map from the induced representation to the space of automorphic forms. The point is that such uniformity, at least for cuspidal Eisenstein series, is provided by the Maass-Selberg relations together with the properties of the intertwining operators at the archimedean places. As for Eisenstein series induced from other discrete spectrum, their properties can be deduced from those of cuspidal Eisenstein series by Langlands' general theory. We remark however, that there is still an

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*Date:* April 3, 2006.

Partially supported by a grant from the Israel Science Foundation.

important difference between the analytic properties of  $K$ -finite Eisenstein series and smooth ones. The former are meromorphic functions of finite order ([Mül00]), while the latter need not be ([Lap06]).

**Acknowledgement.** I would like to thank Hervé Jacquet and Nolan Wallach for encouraging me to write this note and for their comments.

**1.1. Notation.** Let  $G$  be a reductive group over a number field  $F$ . We will often denote  $G(F)$  by  $G$  as well. (Similarly for other groups.) For simplicity of notation we will assume that  $G$  is semisimple. This assumption can be easily lifted. Fix a minimal parabolic subgroup  $P_0$  of  $G$  with a Levi decomposition  $P_0 = M_0U_0$ . Let  $P = MU$  be a parabolic of  $G$  defined over  $F$  containing  $P_0$  such that  $M \supset M_0$ . Let  $T_M$  be the maximal split torus of the center of  $M$ . Thus  $T_M \simeq \mathbb{G}_m^r$  where  $r$  is the co-rank of  $P$  and we let  $A_M$  be the subgroup  $\mathbb{R}_+$  embedded in  $\mathbb{I}_F \simeq T_M(\mathbb{A})$  through the embedding  $\mathbb{R} \hookrightarrow \mathbb{A}_{\mathbb{Q}} \hookrightarrow \mathbb{A}_F$ . Set  $\mathfrak{a}_M^* = X^*(M) \otimes \mathbb{R}$  where  $X^*(\cdot)$  denotes the lattice of characters defined over  $F$ . The dual vector space will be denoted by  $\mathfrak{a}_M$ .

Let  $\delta_P$  be the modulus function of  $P(\mathbb{A})$ . Finally, choose a maximal compact subgroup  $\mathbf{K} = \mathbf{K}_{\infty}\mathbf{K}_f$  of  $G(\mathbb{A})$  which is in a “good position” with respect to  $M_0$  (cf. [MW95]). In particular,  $G(\mathbb{A}) = M(\mathbb{A})U(\mathbb{A})\mathbf{K}$  and  $M(\mathbb{A}) \cap \mathbf{K}$  is a maximal compact of  $M(\mathbb{A})$ . We let  $H = H_M : G(\mathbb{A}) \rightarrow \mathfrak{a}_M$  be the left- $U(\mathbb{A})$  right- $\mathbf{K}$  invariant function on  $G(\mathbb{A})$  so that

$$e^{\langle \chi, H(m) \rangle} = \prod |\chi(m_v)|_v$$

for all  $m = (m_v)_v \in M(\mathbb{A})$  and  $\chi \in X^*(M)$ .

Let  $\mathfrak{S}$  be a locally finite set of affine hyperplanes of  $\mathfrak{a}_{M, \mathbb{C}}^*$  whose vector part is defined over  $\mathbb{R}$ . Let  $\mathcal{P}_{\mathfrak{S}} = \mathcal{P}_{\mathfrak{S}}(\mathfrak{a}_{M, \mathbb{C}}^*)$  be the set of non-zero polynomials on  $\mathfrak{a}_{M, \mathbb{C}}^*$  obtained as products of linear functions, each vanishing on a hyperplane in  $\mathfrak{S}$ . We denote by  $\mathcal{M}_{\mathfrak{S}} = \mathcal{M}_{\mathfrak{S}}(\mathfrak{a}_{M, \mathbb{C}}^*)$  the space of meromorphic functions on  $\mathfrak{a}_{M, \mathbb{C}}^*$  with polynomial singularities in  $\mathfrak{S}$  ([MW95, V.1.2]). Thus  $f \in \mathcal{M}_{\mathfrak{S}}$  if for any  $\lambda \in \mathfrak{a}_{M, \mathbb{C}}^*$  there exist  $R \in \mathcal{P}_{\mathfrak{S}}$  and a neighborhood of  $\lambda$  on which  $Rf$  is holomorphic (or strictly speaking, coincides with a holomorphic function in the complement of  $\cup \mathfrak{S}$ ).

Let  $\mathcal{F}$  be the union of an increasing sequence of Fréchet spaces  $\mathcal{F}_n$  embedded continuously in one another, with the inductive limit topology. We consider  $\mathcal{F}_n$  as a (not necessarily closed) subspace of  $\mathcal{F}$ . We will always assume in addition that  $\mathcal{F}$  is Hausdorff. (This is the case if  $\mathcal{F}$  is the strict inductive limit of the  $\mathcal{F}_n$ 's, but not in general.) In this case, any continuous linear map  $f : V \rightarrow \mathcal{F}$  from a Fréchet space  $V$  factors through a continuous linear map from  $V$  to  $\mathcal{F}_n$  for some  $n$  ([Bou87, I, §3.3, Proposition 1]). Consider the space  $\mathcal{M}_{\mathfrak{S}}(\mathcal{F}) = \mathcal{M}_{\mathfrak{S}}(\mathfrak{a}_{M, \mathbb{C}}^*; \mathcal{F})$  of

meromorphic functions from  $\mathfrak{a}_{M,\mathbb{C}}^*$  to  $\mathcal{F}$  with polynomial singularities in  $\mathfrak{S}$ , which is defined as follows. For every bounded open set  $U \subset \mathfrak{a}_{M,\mathbb{C}}^*$  fix  $R \in \mathcal{P}_{\mathfrak{S}}$  which vanishes on the finitely many hyperplanes in  $\mathfrak{S}$  which intersect  $U$  and let  $\mathcal{M}_{\mathfrak{S}}(U; \mathcal{F})$  be the increasing union of the spaces

$$\{f : U \mapsto \mathcal{F}_n \mid R^n f \text{ is holomorphic}\}$$

with the inductive limit topology, where on each such space we take the semi-norms

$$\sup_{\lambda \in U} |R^n(\lambda) \mu(f(\lambda))|$$

where  $\mu$  is a semi-norm of  $\mathcal{F}_n$ . Note that  $\mathcal{M}_{\mathfrak{S}}(U; \mathcal{F})$  is Hausdorff, because for any  $x \in U \setminus \cup \mathfrak{S}$  the map  $x \mapsto f(x)$  is a continuous map into a Hausdorff space, and these maps separate the points in  $\mathcal{M}_{\mathfrak{S}}(U; \mathcal{F})$ . By definition,  $\mathcal{M}_{\mathfrak{S}}(\mathfrak{a}_{M,\mathbb{C}}^*; \mathcal{F})$  is the space of  $\mathcal{F}$ -valued functions on  $\mathfrak{a}_{M,\mathbb{C}}^*$  whose restriction to any such  $U$  lies in  $\mathcal{M}_{\mathfrak{S}}(U; \mathcal{F})$ . It is equipped with the coarsest topology for which the restriction maps  $\mathcal{M}_{\mathfrak{S}}(\mathcal{F}) \rightarrow \mathcal{M}_{\mathfrak{S}}(U; \mathcal{F})$  are continuous. Note that Cauchy's theorem and integral formula apply to any holomorphic function in  $\mathcal{M}_{\mathfrak{S}}(\mathcal{F})$ , because they hold for holomorphic functions with values in a Fréchet space - cf. [Rud91].

Let  $A_{mod}(G \backslash G(\mathbb{A}))$  be the space of smooth functions on  $G \backslash G(\mathbb{A})$  which are of uniform moderate growth ([MW95, I.2.3]). It is the inductive limit of the Fréchet spaces  $A_{mod}(G \backslash G(\mathbb{A}))_n$  defined by the semi-norms

$$\|f\|_{n,X} = \sup_{g \in \mathfrak{s}} |\delta(X) f(g)| \|g\|^{-n}$$

for any  $X \in U(\mathfrak{g}_{\infty})$  where  $\mathfrak{s}$  is a Siegel set for  $G \backslash G(\mathbb{A})$  and  $\delta$  denotes the action of the universal enveloping algebra  $U(\mathfrak{g}_{\infty})$  of the Lie algebra  $\mathfrak{g}_{\infty}$  of  $G(F_{\infty})$  on  $A_{mod}(G \backslash G(\mathbb{A}))$ . The space  $A_{mod}(G \backslash G(\mathbb{A}))$  is Hausdorff because any point evaluation is a continuous linear form.

## 2. EISENSTEIN SERIES AND INTERTWINING OPERATORS

Let  $\pi$  be a cuspidal automorphic representation of  $M(\mathbb{A})$ . We will always assume that the central character of  $\pi$  is trivial on  $A_M$  and consider  $\pi$  as a subspace of  $L^2(A_M M(F) \backslash M(\mathbb{A}))$ . Let  $\mathcal{A}_P^{\pi}$  denote the space of automorphic forms  $\varphi$  on  $U(\mathbb{A})M \backslash G(\mathbb{A})$  such that for all  $k \in \mathbf{K}$  the function  $m \mapsto \delta_P(m)^{-\frac{1}{2}} \varphi(mk)$  belongs to the space of  $\pi$ . (This differs from the perhaps more common usage of  $\mathcal{A}_P^{\pi}$  where  $m \mapsto \delta_P(m)^{-\frac{1}{2}} \varphi(mk)$  is only required to belong to the  $\pi$ -isotypic part of  $L^2(A_M M(F) \backslash M(\mathbb{A}))$ .) The automorphic realization of  $\pi$  gives rise to an identification of  $\mathcal{A}_P^{\pi}$  with the  $\mathbf{K}$ -finite part of the induced space  $I(\pi) = \text{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \pi$ . Set  $\varphi_{\lambda}(g) = \varphi(g) e^{\langle \lambda, H(g) \rangle}$  for any  $\varphi \in \mathcal{A}_P^{\pi}$ ,  $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$ .

The map  $\varphi \mapsto \varphi_\lambda$  identifies  $I(\pi)$  (as a  $\mathbf{K}$ -module) with any  $I(\pi, \lambda) = I_P(\pi, \lambda) = \text{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \pi \cdot e^{\langle \lambda, H(\cdot) \rangle}$ .

For any  $\varphi \in \mathcal{A}_P^\pi$  consider the Eisenstein series which is the meromorphic continuation of the series

$$E(g, \varphi, \lambda) = E_P(g, \varphi, \lambda) = \sum_{\gamma \in P \backslash G} \varphi_\lambda(\gamma g)$$

which converges for  $\text{Re}(\lambda)$  sufficiently regular in the positive Weyl chamber of  $\mathfrak{a}_M^*$ . Whenever regular,  $\varphi \mapsto E(\varphi, \lambda)$  defines an intertwining map from  $I(\pi, \lambda)_{\mathbf{K}\text{-fin}}$  into the space of automorphic forms on  $G(\mathbb{A})$ .

Exactly as in the archimedean situation (cf. [Wal92, 10.1.1]) the space  $I(\pi)^\infty$  is a Fréchet space with respect to the semi-norms

$$\|X\varphi\|_\infty$$

where

$$\|\varphi\|_\infty = \max_{k \in \mathbf{K}} \|\varphi(k)\|_\pi$$

and  $X$  ranges over the universal enveloping algebra of the complexified Lie algebra  $\mathfrak{k}_\mathbb{C}$  of  $K_\infty$ . Moreover, the argument of [loc. cit] immediately shows the following

**Lemma 1.** *For any  $X \in U(\mathfrak{g}_\mathbb{C})$  there exists  $n \in \mathbb{N}$  and a continuous semi-norm  $\mu$  such that*

$$\|I(X, \pi, \lambda)\varphi\|_\infty \leq (1 + \|\lambda\|)^n \mu(\varphi)$$

for any  $\varphi \in I(\pi)^\infty$  and  $\lambda \in \mathfrak{a}_{M, \mathbb{C}}^*$ . Here  $I(X, \pi, \lambda)$  denotes the action of  $U(\mathfrak{g}_\mathbb{C})$  on  $I(\pi, \lambda)^\infty$  where  $\mathfrak{g}_\mathbb{C}$  is the complexification of the Lie algebra of  $G(\mathbb{R})$ .

Fix an open subgroup  $K_0$  of  $\mathbf{K}_f$  and denote the  $K_0$ -part of a representation  $V$  by  $V^{K_0}$ . Our goal is to prove the following result.

**Theorem 1.** *There exists a locally finite collection of hyperplanes  $\mathfrak{S} = \mathfrak{S}_{K_0}$  such that the map  $\varphi \mapsto E_P(\varphi, \lambda)$  extends to a continuous linear map from  $I(\pi)^{\infty, K_0}$  to  $\mathcal{M}_{\mathfrak{S}}(A_{\text{mod}}(G \backslash G(\mathbb{A})))$ .*

Explicitly, this means that for any compact set  $\Lambda$  of  $\mathfrak{a}_{M, \mathbb{C}}^*$  there exist  $R \in \mathcal{P}_{\mathfrak{S}}$ ,  $n \in \mathbb{N}$  and for each  $X \in U(\mathfrak{g}_\infty)$  a semi-norm  $\mu$  such that

$$\|R(\lambda)\delta(X)E(\varphi, \lambda)\|_n \leq \mu(\varphi)$$

for all  $\varphi \in I(\pi)^{\infty, K_0}$  and  $\lambda \in \Lambda$ . It is enough to check this for  $K_\infty$ -finite sections. (The extension to all sections will be defined by completeness.)

We will prove Theorem 1 below. Let us first mention a corollary thereof. Let  $A_{dec}(G \backslash G(\mathbb{A}))$  be the Fréchet space of functions on  $G \backslash G(\mathbb{A})$  which are rapidly decreasing, with the semi-norms  $\sup_{g \in \mathfrak{s}} f(g) \|g\|^{-n}$ ,  $n = 1, 2, \dots$ . Arthur's truncation defines a continuous linear map from  $A_{mod}(G \backslash G(\mathbb{A}))$  to  $A_{dec}(G \backslash G(\mathbb{A}))$  ([Art80, Lemma 1.4]). We conclude

**Corollary 1.** *The map  $\varphi \mapsto \Lambda^T E(\varphi, \lambda)$  is a continuous linear map from  $I(\pi)^{\infty, K_0}$  to  $\mathcal{M}_{\mathfrak{S}}(A_{dec}(G \backslash G(\mathbb{A})))$ .*

We recall the intertwining operators  $M_{Q|P}(\pi, \lambda)$  as defined for example in [Art82a]. Here  $Q$  belongs to the set  $\mathcal{P}(M)$  of the finitely many parabolic subgroups which contain  $M$  as a Levi subgroup.

These intertwining operators admit local analogues. We first study the local intertwining operators in the archimedean situation. In the following discussion the notation and the objects will pertain to real reductive groups. Recall ([Wal92, Theorem 10.1.5]) that for some  $\rho$  in the positive Weyl chamber there exists a non-zero scalar-valued polynomial  $b(\lambda)$  and a polynomial  $D(\lambda)$  with values in (a finite dimensional subspace of)  $U(\mathfrak{g}_{\mathbb{C}})$  such that

$$(1) \quad b(\lambda) M_{Q|P}(\pi, \lambda) = M_{Q|P}(\pi, \lambda + \rho) I(D(\lambda), \pi, \lambda + \rho).$$

The functions  $b$  and  $D$  depend on  $Q$  (as well as on  $\pi$  of course). However, we can choose  $b$  of the form

$$b(\lambda) = \prod_{\alpha \in \Sigma(P) \cap \Sigma(\bar{Q})} b_{\alpha}(\langle \lambda, \alpha^{\vee} \rangle)$$

for some polynomial functions  $b_{\alpha}$ . For  $\text{Re}(\lambda)$  sufficiently regular in the positive Weyl chamber,  $M_{Q|P}(\pi, \lambda)$  is absolutely convergent and in fact,  $M_{Q|P}(\pi, \lambda)$  is a bounded operator with respect to  $\|\cdot\|_{\infty}$ , independently of  $\lambda$  ([loc. cit., Lemma 10.1.11]). We immediately infer

**Corollary 2.** *Let  $\mathfrak{S}$  consist of the hyperplanes  $\langle \lambda + k\rho, \alpha^{\vee} \rangle = c$  where  $k \in \mathbb{N}$  and  $c$  is a root of  $b_{\alpha}$ . Then all matrix coefficients of  $M_{Q|P}(\pi, \lambda)$  lie in  $\mathcal{M}_{\mathfrak{S}}$ . Moreover, for  $\lambda$  in a compact set, there exists a semi-norm  $\mu$  and  $R \in \mathcal{P}_{\mathfrak{S}}$  such that*

$$(2) \quad \|R(\lambda) M_{Q|P}(\pi, \lambda) \varphi\| \leq \mu(\varphi)$$

for all  $\varphi \in I(\pi)^{\infty}$ .

In fact,  $R$  and  $\mu$  can be chosen uniformly on any cone  $\langle \lambda, \alpha^{\vee} \rangle > d$ ,  $\forall \alpha \in \Delta_P$ .

Back to the global setup, we can write, for each  $Q \in \mathcal{P}(M)$ , the restriction of  $M_{Q|P}(\pi, \lambda)$  to  $I(\pi)^{\infty, K_0}$  as

$$m_{\pi, Q|P}(\lambda) \prod_{v \in S} M_{Q|P, v}(\pi_v, \lambda)$$

where  $S$  is a finite set of places, depending on  $K_0$ , containing all the archimedean ones. The function  $m_{\pi, Q|P}$  belongs to  $\mathcal{M}_{\mathfrak{S}}$  for a suitable  $\mathfrak{S}$ . For every finite  $v \in S$   $M_{v, Q|P}(\pi_v, \lambda)$  is a rational function in  $q_v^{\langle \lambda, \alpha^\vee \rangle}$ ,  $\alpha \in \Sigma_P$ . We conclude that for a suitable  $\mathfrak{S}$ , all matrix coefficients of  $M_{Q|P}(\pi, \lambda)$  (as an operator on  $I(\pi)^{K_0}$ ) lie in  $\mathcal{M}_{\mathfrak{S}}$ , and the relation (2) immediately carries over to the global setting. Also, by ([MW95, Remark IV.4.4]) for any  $\varphi \in I(\pi)_{\mathbf{K}-fin}^{K_0}$ ,  $E(\varphi, \lambda) \in \mathcal{M}_{\mathfrak{S}}(L_{loc}^2(G \backslash G(\mathbb{A})))$ .

In order to prove Theorem 1 we argue as in [Lap06, §6]. Consider the truncated Eisenstein series  $\Lambda^T E(g, \varphi, \lambda)$  (cf. [Art80, §4]). The main step will be the following upper bound for the  $L^2$ -norm of  $\Lambda^T E(\cdot, \varphi, \lambda)$ . For the rest of the section we assume that  $\lambda$  is confined to a compact set which will be fixed throughout.

**Proposition 1.** *There exists an element  $R \in \mathcal{P}_{\mathfrak{S}}$ , a semi-norm  $\mu$  on  $I(\pi)$  and  $C \geq 1$  such that*

$$(3) \quad \|R(\lambda)\Lambda^T E(\cdot, \varphi, \lambda)\|_{L^2(G \backslash G(\mathbb{A}))} \leq C^{\|T\|} \mu(\varphi)$$

for all  $\varphi \in I(\pi)^\infty$  and all sufficiently regular  $T$ . More generally, for any  $X \in U(\mathfrak{g}_{\infty, \mathbb{C}})$  a similar upper bound (with  $\mu$  depending on  $X$ ) holds for  $\|R(\lambda)\Lambda^T E(\cdot, I(X, \pi, \lambda)\varphi, \lambda)\|_{L^2(G \backslash G(\mathbb{A}))}$ .

*Proof.* First note that the second part follows from the first in view of Lemma 1. Set

$$\|\Lambda^T E(\cdot, \varphi, \lambda)\|_{L^2(G \backslash G(\mathbb{A}))}^2 = (\Omega^T(\lambda)\varphi, \varphi).$$

By the Maass-Selberg relations, the operator  $\Omega^T(\lambda)$  is given by the sum over the representatives  $s \in W/W_M$  such that  $sMs^{-1} = M$  of the value at  $\lambda' = \lambda$  of

$$\sum_{Q \in \mathcal{P}(M_P)} M_{Q|P}(\lambda)^* M_{Q|P}(s\lambda') M_{P|P}(s, \lambda') e^{\langle s\lambda' + \bar{\lambda}, Y_Q(T) \rangle} \theta_Q(s\lambda' + \bar{\lambda})^{-1}$$

for some affine function  $Y_Q$  of  $T$  (cf. [Art82a, p. 1295-6], where  $Y_Q(T)$  is described explicitly.) Unlike in [loc. cit.], we do not assume that  $\lambda \in i\mathfrak{a}_M^*$ . Recall also ([loc. cit., p. 1310]) the  $(G, M)$ -families (in  $\Lambda$ )

$$\mathcal{M}_Q(\lambda, \Lambda) = M_{Q|P}(\lambda)^* M_{Q|P}(-\bar{\lambda} + \Lambda)$$

$$c_Q(T, \Lambda) = e^{\langle \Lambda, Y_Q(T) \rangle}$$

$$\mathcal{M}_Q^T(\lambda, \Lambda) = c_Q(T, \Lambda) \mathcal{M}_Q(\lambda, \Lambda).$$

Then  $\Omega^T(\lambda)$  is the sum over  $s$  of the value at  $\lambda' = \lambda$  of

$$\sum_{Q \in \mathcal{P}(M_P)} \mathcal{M}_Q(\lambda, s\lambda' + \bar{\lambda}) c_Q(T, s\lambda' + \bar{\lambda}) M_{P|P}(s, \lambda') \theta_Q(s\lambda' + \bar{\lambda})^{-1}$$

which in the notation of [loc. cit.] is  $\mathcal{M}_M^T(\lambda, s\lambda + \bar{\lambda})M_{P|P}(s, \lambda)$ . The global analogue of Corollary 2 applies to  $M_{P|P}(s, \lambda)$  (on  $I(\pi)^{\infty, K_0}$ ). It remains to show that for  $\lambda, \Lambda$  in a compact set there exist  $r \in \mathbb{N}$ ,  $R \in \mathcal{P}_{\mathfrak{S}}$ ,  $C \geq 1$  and for any  $X$  a semi-norm  $\mu$  such that

$$(4) \quad \|R(\lambda)R(-\bar{\lambda} + \Lambda)\mathcal{M}_M^T(\lambda, \Lambda)I(X, \pi, \lambda)\varphi\| \leq \mu(\varphi)C^{\|T\|}$$

for all  $\varphi$ . Fix a vector  $\xi \in \mathfrak{a}_M^*$  such that  $\langle \xi, \alpha^\vee \rangle \neq 0$  for all  $\alpha \in \Sigma(T_M, G)$ . Write  $R(\lambda)R(-\bar{\lambda} + \Lambda)\mathcal{M}_M^T(\lambda, \Lambda)$  as a Cauchy integral

$$\oint \frac{R(\lambda)R(-\bar{\lambda} + \Lambda + z\xi)\mathcal{M}_M^T(\lambda, \Lambda + z\xi)}{z} dz$$

over a circle  $C_r$  of radius  $r$  centered at 0 (for an appropriate  $R \in \mathcal{P}_{\mathfrak{S}}$ ). If  $r$  is sufficiently large with respect to  $\lambda$  and  $\Lambda$ , the denominators  $\theta_Q(\Lambda + z\xi)$  appearing in the expression for the integrand are bounded away from zero. The dependence on  $T$  is controlled by  $c_Q$ . To conclude (4) it remains to apply the global analogue of (2).  $\square$

Fix a compact set  $B \subset G(\mathbb{A})$  which is left and right  $\mathbf{K}$ -invariant. We first prove that there exists  $n \in \mathbb{N}$  and a semi-norm  $\mu$  such that

$$(5) \quad \|E(I(f, \pi, \lambda)\varphi, \lambda)\|_n \leq \|f\|_\infty \mu(\varphi)$$

for all bi- $K$ -finite  $f$  supported in  $B$  and  $K$ -finite  $\varphi$ . Indeed, for  $T$  regular enough (depending only on  $B$  and  $g$ ) we have

$$\begin{aligned} E(g, I(f, \lambda)\varphi, \lambda) &= \int_{G(\mathbb{A})} f(g^{-1}x)E(x, \varphi, \lambda) dx \\ &= \int_{G(\mathbb{A})} f(g^{-1}x)\Lambda^T E(x, \varphi, \lambda) dx. \end{aligned}$$

In fact, we can choose  $T$  so that  $\|T\|$  is bounded by a constant multiple of  $1 + \log\|g\|$  (cf. [Lap06, Lemma 6.2]). We rewrite the above as

$$\int_{G \backslash G(\mathbb{A})} \left( \sum_{\gamma \in G} f(g^{-1}\gamma x) \right) \Lambda^T E(x, \varphi, \lambda) dx.$$

By [MW95, I.2.4] for some  $r$  and  $c$  (depending on  $B$ ) we have

$$\left| \sum_{\gamma \in G} f(g^{-1}\gamma x) \right| \leq c\|g\|^r \|f\|_\infty$$

for all  $x, g \in G(\mathbb{A})$ . Hence, by Cauchy-Schwartz

$$|E(g, I(f, \pi, \lambda)\varphi, \lambda)| \leq c'\|f\|_\infty \|g\|^r \|\Lambda^T E(\cdot, \varphi, \lambda)\|_2.$$

We conclude (5) from Proposition 1.

More generally, for any  $X \in U(\mathfrak{g}_{\infty, \mathbb{C}})$  we have a similar bound for  $E(I(f, \pi, \lambda)I(X, \pi, \lambda)\varphi, \lambda)$ .

Thus, the map  $f \rightarrow E(I(f, \pi, \lambda)\varphi, \lambda)$  extends to  $C_c(G(\mathbb{A}))$ .

By [Art78, §4] there exist  $f_1 \in C_c^\infty(G(F_\infty))$ ,  $f_2 \in C_c(G(F_\infty))$  and  $Z \in U(\mathfrak{g}_{\infty, \mathbb{C}})$  such that  $f_1 + f_2 \star Z$  is equal to the Dirac distribution at the identity. (In fact, we can choose  $f_2 \in C_c^m(G(F_\infty))$  for any given  $m$ .) Hence, if  $F_i = \text{vol}(K_0)^{-1} \cdot f_i \otimes 1_{K_0}$ ,  $i = 1, 2$  where  $1_{K_0}$  is the characteristic function of  $K_0$  then we have

$$E(\varphi, \lambda) = E(I(F_1, \lambda)\varphi, \lambda) + E(I(F_2, \lambda)I(Z, \lambda)\varphi, \lambda).$$

It follows that there exist  $R \in \mathcal{P}_{\mathfrak{S}}$ ,  $n \in \mathbb{N}$  and a semi-norm  $\mu$  such that

$$\|R(\lambda)E(\varphi, \lambda)\|_n \leq \mu(\varphi).$$

A similar estimate (with  $\mu$  depending on  $X$ ) holds for  $\delta(X)E(\varphi, \lambda)$ . This concludes the proof of Theorem 1.

*Remark 1.* Note that in the case  $P = G$  the above argument shows that the map from the smooth part of  $\pi$  to  $A_{\text{mod}}(G \backslash G(\mathbb{A}))$  (and therefore to  $A_{\text{dec}}(G \backslash G(\mathbb{A}))$ ) is continuous. Of course, this also follows from the automatic continuity theorem (which we never used).

### 3. NON-CUSPIDAL EISENSTEIN SERIES

Next, we generalize Theorem 1 to Eisenstein series induced from any representation  $\pi$  in the discrete spectrum of  $M \backslash M(\mathbb{A})^1$ . By Langlands' theory  $E_P(\varphi, \lambda)$  is given in terms of residues of cuspidal Eisenstein series. We briefly recall how this is done. (See [MW95, Ch. VI] for a precise statement.) First, recall the notion of *residue data* defined in [MW95, V.1.3]. It is a linear map (depending on some choices)  $\text{Res}_{V'} : \mathcal{M}_{\mathfrak{S}}(\mathfrak{a}_{M, \mathbb{C}}^*; \mathcal{F}) \rightarrow \mathcal{M}_{\mathfrak{S}'}(V'; \mathcal{F})$  where  $V'$  is an intersection of hyperplanes in  $\mathfrak{S}$  and

$$\mathfrak{S}' = \{\sigma \cap V' : \sigma \in \mathfrak{S}, \sigma \not\supset V'\}.$$

Note that  $\text{Res}_{V'}$  is continuous. This amounts to showing that for any small open  $U \subset \mathfrak{a}_{M, \mathbb{C}}^*$  and  $R \in \mathcal{P}_{\mathfrak{S}}$  there exists a small open  $U' \subset U \cap V'$ ,  $R' \in \mathcal{P}_{\mathfrak{S}'}$  and a constant  $c$  such that for any semi-norm  $\mu$  of  $\mathcal{F}_n$ ,  $n \in \mathbb{N}$  we have

$$\sup_{U'} \mu(R'(\lambda') \text{Res}_{V'} f(\lambda')) \leq c \sup_U \mu(R(\lambda)f(\lambda))$$

for any  $f \in \mathcal{M}_{\mathfrak{S}}(\mathcal{F})$  such that the restriction of  $Rf$  to  $U$  is holomorphic with values in  $\mathcal{F}_n$ . This immediately follows from the discussion of [loc. cit.] and Cauchy's integral formula. It will be useful to consider also the space of functions with polynomial singularities on a given open set of  $\mathfrak{a}_{M, \mathbb{C}}^*$ . A similar statement holds for taking residues in this setup.

Let  $B$  a parabolic subgroup of  $G$  and  $\sigma$  a cuspidal representation of its Levi part. Fix  $R \gg 0$  and consider the Fréchet space  $\mathcal{PW}^R(\mathfrak{a}_{B,\mathbb{C}}^*; I(\sigma)^\infty)$  consisting of holomorphic functions  $\varphi$  on the tube  $\mathfrak{T} = \mathfrak{T}_R = \{\lambda \in \mathfrak{a}_{B,\mathbb{C}}^* : \|\operatorname{Re} \lambda\| < R\}$  with values in  $I(\sigma)^\infty$  such that the norms

$$\sup_{\mathfrak{T}} \|I(X, \sigma, \lambda)\varphi(\lambda)\|_\infty (1 + \|\lambda\|)^n, n = 1, 2, \dots, X \in U(\mathfrak{g}_{\mathbb{C}})$$

are finite. (It is enough to take  $X \in U(\mathfrak{k}_{\mathbb{C}})$  by Lemma 1.) This space has an action of  $G(\mathbb{A})$  given by

$$g\varphi(\lambda) = I(g, \sigma, \lambda)\varphi(\lambda).$$

It is easy to see that  $\mathcal{PW}(\mathfrak{a}_{B,\mathbb{C}}^*; I(\sigma)^\infty)$  is of *moderate growth* in the sense of [Wal92, 11.5.1]. The argument is similar to that of Lemma 11.5.1 of [loc. cit.]. Using Theorem 1 we obtain a continuous linear map  $\varphi \mapsto E_B(\varphi(\lambda), \lambda)$  from  $\mathcal{PW}^R(\mathfrak{a}_{B,\mathbb{C}}^*; I(\sigma)^\infty)$  to  $\mathcal{M}_{\mathfrak{S}}(\mathfrak{T}; A_{\operatorname{mod}}(G \backslash G(\mathbb{A})))$ . A similar statement holds for the Eisenstein series  $E_B^P(\varphi(\Lambda), \Lambda)$ ,  $\Lambda \in (\mathfrak{a}_B^P)_{\mathbb{C}}^*$  where in its defining series, the sum is taken over  $\gamma \in B \backslash P$ .

Let  $\mathcal{A}_P^\pi$  be the space of automorphic forms  $\varphi$  on  $U(\mathbb{A})M \backslash G(\mathbb{A})$  such that for all  $k \in \mathbf{K}$ ,  $m \mapsto \delta_P(m)^{-\frac{1}{2}}\varphi(mk)$  belongs to the space of  $\pi$ . We identify it with the  $\mathbf{K}$ -finite part of  $I(\pi)$ . If  $\chi$  is the cuspidal data pertaining to  $\pi$  then  $\mathcal{A}^\pi$  is contained in the space  $\mathcal{A}_{P,\chi}^2$  of automorphic forms on  $U(\mathbb{A})M \backslash G(\mathbb{A})$  having cuspidal data  $\chi$  such that  $\varphi(ag) = \delta_P(a)^{\frac{1}{2}}\varphi(g)$  for all  $a \in A_M$  and  $\int_{U(\mathbb{A})MA_M \backslash G(\mathbb{A})} |\varphi(g)|^2 dg < \infty$ . Fix  $K_0$  as before and consider the space

$$\mathcal{PW}_\chi^R = \bigoplus_{(B,\sigma) \in \chi, B \subset P} \mathcal{PW}^R((\mathfrak{a}_B^P)_{\mathbb{C}}^*, I(\sigma)^{\infty, K_0})$$

and its subspace  $\mathcal{PW}_{\chi, K\text{-fin}}^R$  of  $\mathbf{K}$ -finite vectors. By [MW95] (cf. [Art82b, §2])  $\mathcal{A}_{P,\chi}^{2, K_0}$  is the image of the map

$$\mathfrak{e}^P : \mathcal{PW}_{\chi, K\text{-fin}}^R \rightarrow A_{\operatorname{mod}}(U(\mathbb{A})M \backslash G(\mathbb{A}))^{K_0}$$

which is given by the sum of residue data of cuspidal Eisenstein series  $E_B^P(\varphi(\Lambda), \Lambda)$  ( $\Lambda \in (\mathfrak{a}_B^P)_{\mathbb{C}}^*$ ) at certain points. This map is also given by a spectral projection applied to the pseudo-Eisenstein series built from  $\varphi$ . It is therefore an intertwining map. Furthermore, by the above, it extends to a continuous linear map, denoted  $\overline{\mathfrak{e}}^P$ , on  $\mathcal{PW}_\chi^R$ . By [Wal92, Theorem 11.8.2] the image of  $\overline{\mathfrak{e}}^P$  contains the smooth part of  $I(\pi)^{K_0}$ , because it contains a dense subspace thereof. By [Bou87, II, §4.7, Proposition 12] we can find a continuous (not necessarily linear) map

$$\iota : I(\pi)^{\infty, K_0} \rightarrow \mathcal{PW}_\chi^R$$

such that  $\overline{\mathfrak{e}}^P \circ \iota = Id$ .

Similarly, we have a continuous linear map

$$\mathfrak{e} : \mathcal{PW}_{\chi}^R \rightarrow \mathcal{M}_{\mathfrak{S}}(\mathfrak{a}_{M,\mathbb{C}}^*; A_{\text{mod}}(G \backslash G(\mathbb{A}))^{K_0}),$$

given by the sum over residue data of the cuspidal Eisenstein series  $E_B(\varphi(\Lambda), \lambda + \Lambda)$  ( $\Lambda \in (\mathfrak{a}_B^P)_{\mathbb{C}}^*$ ,  $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$ ). We have

$$\mathfrak{e}(\varphi, \lambda) = E_P(\mathfrak{e}^P(\varphi), \lambda)$$

on  $\mathcal{PW}_{\chi, \mathbf{K}\text{-fin}}^R$ . In other words, on  $\mathcal{PW}_{\chi, \mathbf{K}\text{-fin}}^R$   $\mathfrak{e}$  factors through  $\mathcal{A}_{P,\chi}^{2,K_0}$  as the map  $E_P$ . The map  $\mathfrak{e} \circ \iota$  is a continuous map from  $I(\pi)^{\infty, K_0}$  to  $A_{\text{mod}}(G \backslash G(\mathbb{A}))^{K_0}$  which extends  $E_P$ . In particular, it is linear.

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