Notes on the adèles

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The field of rational numbers $\mathbb{Q}$ has the following valuations. The "usual" archimedean absolute value $|\cdot|_\infty$, and for each prime p the non-archimedean valuations:

$$|p^k \frac{m}{n}|_p = p^{-k} \text{ for } p \not| mn$$

It satisfies

$$|x+y|_p \leq \max(|x|_p, |y|_p)$$

These are all the valuation up to equivalence. Let $\mathbb{Q}_p$ be the completion with respect to $|\cdot|_p$. It is the field of p-adic numbers. We also set $\mathbb{Q}_\infty = \mathbb{R}$. ($|\cdot|_p$ is the modulus function with respect to a Haar measure on $\mathbb{Q}_p$.) For $p = \infty$ $\mathbb{Z}$ is closed, but if $p < \infty$ the closure of $\mathbb{Z}$, denoted by $\mathbb{Z}_p$ or $\mathcal{O}_p$, is a maximal subring of $\mathbb{Q}_p$. It is the ring of p-adic integers. It is a local (commutative) ring whose maximal ideal is $p\mathbb{Z}_p$. We have

$$\mathbb{Z}_p/p\mathbb{Z}_p \simeq \mathbb{Z}/p\mathbb{Z}$$

and $\mathbb{Z}_p = \varprojlim \mathbb{Z}_p/p^n\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$. It is a compact totally disconnected group. The invertible elements are

$$\mathbb{Z}_p^* = \{x \in \mathbb{Q}_p : |x| = 1\}.$$

We have

$$\mathbb{Q}_p = \bigcup_{n \in \mathbb{Z}} p^n \mathbb{Z}_p$$

and

$$\bigcap_{n \in \mathbb{Z}} p^n \mathbb{Z}_p = 0$$

It follows that

$$\mathbb{Q}_p/\mathbb{Z}_p = \varinjlim \mathbb{Z}/p^n\mathbb{Z}$$

Any $x \in \mathbb{Q}_p$ can be written as $\frac{a}{p^n} + b$ where $a \in \mathbb{Z}$ and $b \in \mathbb{Z}_p$. Define

$$\psi_p(x) = e^{\frac{2\pi i a}{p^n}}$$

This is a continuous (locally constant) character of $\mathbb{Q}_p$, trivial on $\mathbb{Z}_p$. The character group of $\mathbb{Q}_p$ is given by

$$\psi_p(a\cdot) : a \in \mathbb{Q}_p$$

Thus $\widehat{\mathbb{Q}_p} \simeq \mathbb{Q}_p$, although not canonically. We also have of course $\widehat{\mathbb{R}} \simeq \mathbb{R}$ non-canonically. We let

$$\psi_\infty(x) = e^{-2\pi i x}$$

The ring of adèles $\mathbb{A}_Q$ is defined by the restricted product

$$\mathbb{A} = \{(x_p)_{p\leq\infty} : x_p \in \mathbb{Q}_p \text{ for all } p \text{ and } x_p \in \mathbb{Z}_p \text{ for almost all } p\}$$

It is a locally compact topological ring. The basis for the topology is

$$U \times \prod_{p\notin S} \mathbb{Z}_p$$

where U is an open neighborhood of $\prod_{p\in S} \mathbb{Q}_p$ (S is a finite set of primes). Define $\psi = \otimes\psi_p$. It is a continuous character of $\mathbb{A}$. We have

$$\widehat{\mathbb{A}} \simeq \mathbb{A}$$

the isomorphism given by

$$a \mapsto \psi_a = \psi(a\cdot)$$

We can embed $\mathbb{Q}$ diagonally in $\mathbb{A}$. It is discrete, and a (set theoretic) fundamental domain is

$$[0,1) \times \prod_{p<\infty} \mathbb{Z}_p$$

so $\mathbb{A}/\mathbb{Q}$ is compact (but very different topologically or algebraically from $\mathbb{T} \times \prod_{p<\infty} \mathbb{Z}_p$). Clearly, ψ is trivial on $\mathbb{Q}$. Thus, ψ_a is trivial on $\mathbb{Q}$ for $a \in \mathbb{Q}$.

Conversely, if ψ_a is trivial on $\mathbb{Q}$ then $a \in \mathbb{Q}$. Thus $\widehat{\mathbb{A}/\mathbb{Q}} \simeq \mathbb{Q}$, with $\mathbb{Q}$ discrete. It follows that $\widehat{\mathbb{Q}} = \mathbb{A}/\mathbb{Q}$. Since

$$\mathbb{Q} = \varinjlim \frac{1}{n}\mathbb{Z}$$

we get

$$\mathbb{A}/\mathbb{Q} = \varprojlim_n \mathbb{T}, \text{ a solonoid}$$

where $\mathbb{T}$ is a torus and the maps are given by taking powers. Thus $\mathbb{A}/\mathbb{Q}$ is connected.

Exercise: A locally compact group is connected if and only if it doesn't have a non-trivial open subgroup. Thus, an Abelian locally compact group is connected if and only if its Pontriagin dual has no compact subgroups.

The ideles $\mathbb{I}_Q$ (which historically were introduced earlier by Chevalley) are the multiplicative group of $\mathbb{A}$, i.e.

$$\mathbb{I} = \{(x_p)_{p\leq\infty} : x_p \in \mathbb{Q}_p^* \text{ for all } p \text{ and } x_p \in \mathbb{Z}_p^* \text{ for almost all } p\}$$

It is a locally compact topological group where the topology is given by

$$U \times \prod_{p\notin S} \mathbb{Z}_p^*$$

where U is an open neighborhood of $\prod_{p\in S} \mathbb{Q}_p^*$, S as before. This is **N**OT the induced topology from $\mathbb{A}$. We write $\mathbb{I}^f = \prod'_{p<\infty} \mathbb{Q}_p^*$. Define

$$|\cdot| : \mathbb{I} \to \mathbb{R}_+$$

by

$$|x| = \prod_p |x|_p$$

Let $\mathbb{I}^1$ be the kernel. Clearly, $\mathbb{Q}^* \subset \mathbb{I}^1$ and in fact, $\mathbb{Q}^*$ is a discrete subgroup. A fundamental domain of $\mathbb{Q}^*$ in $\mathbb{I}$ is

$$\mathbb{R}_+ \times \prod_{p<\infty} \mathbb{Z}_p^*$$

(unique factorization). We have,

$$\mathbb{I}^1/\mathbb{Q}^* = \prod_{p<\infty} \mathbb{Z}_p^* = \varprojlim (\mathbb{Z}/n\mathbb{Z})^* = \mathbb{Z}^*$$

By Kronecker-Weber Theorem

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})^{ab} = \varprojlim \mathrm{Gal}(\mathbb{Q}(e^{\frac{2\pi i}{n}})/\mathbb{Q}) = \varprojlim (\mathbb{Z}/n\mathbb{Z})^* = \mathbb{Z}^*$$

This "coincidence" is class field theory for $\mathbb{Q}$.

If F is a number field, the ring of adèles $\mathbb{A}_F$ of F is defined by $\mathbb{A}_F = \mathbb{A} \otimes_{\mathbb{Q}} F$. It can also be defined in terms of completions of F. The non-archimedean valuations of F are defined in terms of the prime ideals of the ring of integers $\mathcal{O}_F$ of F. $\mathbb{A}_F$ is locally compact and F imbeds discretely and co-compactly in $\mathbb{A}_F$. If we take the Haar measure $dx = \otimes_v dx_v$ such that dx_v has volume 1 on $\mathcal{O}_v$ and is the standard Lebesgue measure in the archimedean places then $\mathrm{vol}(F\backslash\mathbb{A}_F) = |D_F|^{\frac{1}{2}}$ where D_F is the discriminant of F. The ideles $\mathbb{I}_F$ are defined as the multiplicative group of $\mathbb{A}_F$. We can write $\mathbb{I}_F = \mathbb{I}_F^\infty \times \mathbb{I}_F^f$. We have

$$|\cdot| : \mathbb{I}_F \to \mathbb{R}_+$$

with kernel $\mathbb{I}_F^1$. F^* imbeds discretely in $\mathbb{I}_F^1$ and $F^*\backslash\mathbb{I}_F^1$ is compact. On $\mathbb{I}_F$ we can take the Haar measure $\otimes n_v \frac{dx_v}{|x_v|_v}$ where

$$n_v = \begin{cases} (1-1/q_v)^{-1} & v \text{ non-archimedean of residue field size } q_v \\ 1 & \text{otherwise} \end{cases}$$

The volume of $F^*\backslash\mathbb{I}_F^1$ is then $\lambda_{-1} = \mathrm{res}_{s=1}\zeta_F(s)$. The maximal subgroup K of $\mathbb{I}_F^f$ is $\prod \mathcal{O}_v^*$. The class group of F is isomorphic to $F^*\backslash\mathbb{I}_F^f/K$. We have

$$\mathcal{O}_F^*\backslash(\mathbb{I}_F^\infty)^1 \to F^*\backslash\mathbb{I}_F^1/K \to F^*\backslash I_F^f/K$$

so the finite volume of $F^*\backslash\mathbb{I}_F^1$ captures both the finiteness of the class number and Dirichlet's theorem on the rank of $\mathcal{O}_F^*$.

Hecke characters are characters of $F^*\backslash\mathbb{I}_F^1$. They are just Dirichlet characters if $F = \mathbb{Q}$. In general, the classical Dirichlet characters are the Hecke characters of finite order. An infinite order Hecke character for $F = \mathbb{Q}(i)$ is given by

$$\chi((x_v)_v) = \chi_\infty(x_\infty)\prod_{\mathcal{P}}\chi_{\mathcal{P}}(x_{\mathcal{P}})$$

where

$$\chi_\infty(x_\infty) = (\frac{x_\infty}{|x_\infty|})^4$$

$\chi_{\mathcal{P}}$ is the unramified character with $\chi_{\mathcal{P}}(\mathcal{P}) = p^2/\mathcal{P}^4$ if $\mathcal{P}\overline{\mathcal{P}}$ is the prime $p \in \mathbb{Z}$ and $\chi_p \equiv 1$ is p is inert.

If G is an algebraic group over F we let $G(\mathbb{A}_F)$ be the group of points over $\mathbb{A}_F$. The topology is inherited from $GL_n(\mathbb{A}_F) = \prod' GL_n(F_v)$, restricted product with respect to $GL_n(\mathcal{O}_v)$. $G(\mathbb{A}_F)$ is locally compact topological group which has $G(F)$ as discrete group. If G is semisimple then $G(F)\backslash G(\mathbb{A}_F)$ is of finite volume.

GL_2 case: Let f be a modular form of weight k with respect to

$$\Gamma(N) = \{\gamma \in SL_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N\}.$$

(Soon, we will work with

$$\Gamma_0(N) = \{\gamma \in SL_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N\}$$

and a character χ of $(\mathbb{Z}/N\mathbb{Z})^*$.) We can lift f to a function on $\Gamma(N)\backslash SL_2(\mathbb{R})$ by $\varphi_f(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) = (ci+d)^{-k} f(\frac{ai+b}{ci+d})$. Then $K = SO(2)$ acts on the right on φ_f by the character indexed by k. Let

$$K(N) = \prod_{p<\infty} \{g \in SL_2(\mathbb{Z}_p) : g \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N_p}\}$$

where N_p is the highest p-power dividing N. $K(N)$ is an open compact subgroup of $SL_2(\mathbb{A})^f$. We have

$$SL_2(\mathbb{Q})\backslash SL_2(\mathbb{A})/K(N) \leftrightarrow \Gamma(N)\backslash SL_2(\mathbb{R})$$

i.e. $SL_2(\mathbb{A}) = SL_2(\mathbb{Q}) SL_2(\mathbb{R}) K(N)$ (strong approximation). In other words

$$SL_2(\mathbb{Q})\backslash SL_2(\mathbb{A}) \leftrightarrow \varprojlim \Gamma(N)\backslash SL_2(\mathbb{R})$$

At this stage it is better to work with modular forms for $\Gamma_0(N)$ and to pass to GL_2. Defining subgroups $K_0(N)$ of $GL_2(\mathbb{A})^f$ we have

$$Z(\mathbb{A}) GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})/K_0(N) \leftrightarrow \Gamma_0(N)\backslash SL_2(\mathbb{R}) \tag{1}$$

where $Z(\mathbb{A}) \simeq \mathbb{I}_{\mathbb{Q}}$ is the center of $GL_2(\mathbb{A})$ and we can view φ_f as a right-$K_0(N)$-invariant function on $GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}_F)$. (If χ appears, it can viewed as the central character.) We can look at the representation generated by φ_f. The same can be done for Maass forms. (For general number fields or other discrete subgroups like $\Gamma(N)$ the analogue of (1) is not quite true. One has to take finitely many copies on the right hand side.)

The Hecke operators can be viewed as the action of the matrix $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ for $p \not| N$. More precisely it is the coset

$$T_p = GL_2(\mathbb{Z}_p) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} GL_2(\mathbb{Z}_p) = \bigcup \begin{pmatrix} p & a \\ 0 & 1 \end{pmatrix} GL_2(\mathbb{Z}_p) \bigcup \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} GL_2(\mathbb{Z}_p)$$

More generally, if π is an invariant subspace of functions on $GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})$ with respect to the right regular representation of $GL_2(\mathbb{A})$ then for any p the ("big") Hecke algebra of all compactly supported (modulo the center) locally constant functions on $GL_2(\mathbb{Q}_p)$ under convolution, acts on π. For different p's these actions commute and we get an action of the tensor product of the Hecke algebras. The fixed part under $GL_2(\mathbb{Z}_p)$ is invariant under the "classical" Hecke algebra of bi-$GL_2(\mathbb{Z}_p)$-invariant functions. The latter is commutative and in fact a polynomial algebra in one variable (T_p). (There is also an appropriate notion of a Hecke algebra at the archimedean place.) Note that the multiplicity may be bigger if we just restrict ourselves to the $GL_2(\mathbb{R})$ action. If we consider all places, the multiplicity is one. Also, one can prove that any (admissible) irreducible representation of $GL_2(\mathbb{A})$ factors as a tensor product of irreducible representations of $GL_2(\mathbb{Q}_p)$.

reduction theory for $GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})$: Fix (compact) fundamental domains ω_+, ω_* of $\mathbb{Q}\backslash\mathbb{A}_Q$ and $\mathbb{Q}^*\backslash\mathbb{I}_Q^1$. For $t_0 > 0$ we define the Siegel set

$$\mathcal{S} = \Omega \cdot A_0(t_0) \cdot K \subset GL_2(\mathbb{A})$$

where

$$\Omega = \{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \omega_*, c \in \omega_+ \}$$

(this is a fundamental domain for the upper diagonal matrices),

$$A_0(t_0) = \{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{R}_+ : \frac{a}{b} > t_0 \}$$

(this is the only non-compact part) and

$$K = SO(2) \times \prod_{p<\infty} GL_2(\mathbb{Z}_p)$$

is (the standard) maximal compact of $GL_2(\mathbb{A})$. Then for $t_0 > 0$ sufficiently small

$$GL_2(\mathbb{A}_F) = GL_2(\mathbb{Q}) \cdot \mathcal{S}$$

and $\{\gamma \in GL_2(\mathbb{Q}) : \gamma\mathcal{S} \cap \mathcal{S} \neq \emptyset\}$ is finite. In other words, there is only one cusp.

Notes about bibliography

The $GL(1)$ theory, including Tate's thesis is explained in [Lan94], [GGPS90] and [RV99]. The $GL(2)$ theory, including the passage from classical language to representation theory and adèles is explained in [Gel75], [Bum97], [PS79a]. More advanced texts are the Corvalis and the Boulder volumes especially the articles [BJ79], [Bor79], [Bor66a], [Bor66b], [Fla79], [PS79b]. The book [MW95] of Mœglin-Waldspurger elaborates on Langlands' work [Lan76].

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