Weyl Group Multiple Dirichlet Series II: The Stable Case

Ben Brubaker[1], **Daniel Bump**[1], **Solomon Friedberg**[2]

[1] Department of Mathematics, Stanford University, Stanford CA 94305-2125, USA
[2] Department of Mathematics, Boston College, Chestnut Hill, MA 02467-3806, USA

The date of receipt and acceptance will be inserted by the editor

Abstract. To each reduced root system Φ of rank r, and each sufficiently large integer n, we define a family of multiple Dirichlet series in r variables, whose group of functional equations is isomorphic to the Weyl group of Φ. The coefficients in these Dirichlet series exhibit a multiplicativity that reduces the specification of the coefficients to those that are powers of a single prime p. For each p, the number of nonzero such coefficients is equal to the order of the Weyl group, and each nonzero coefficient is a product of n-th order Gauss sums. The root system plays a basic role in the combinatorics underlying the proof of the functional equations.

1. Introduction

Let $\Phi \subset \mathbb{R}^r$ be a reduced root system of rank r, and n an integer > 1, and let F be an algebraic number field that contains the group μ_n of n-th roots of unity; we assume also that -1 is an n-th power in F, which implies that it is totally complex. With these data we described in [3] a family of Dirichlet series in several variables, to be called *Weyl group multiple Dirichlet series*. Their coefficients involve n-th order Gauss sums. In this paper we will give a complete proof of the analytic continuation and functional equations of these multiple Dirichlet series subject to the *stability assumption*, which states that n is sufficently large, depending on Φ. The stability assumption is stated precisely in Section 3.

We begin by recalling the definition of the Gauss sums which appear in these Dirichlet series.

Let S be a finite set of places of F containing the archimedean ones, all those which are ramified over $\mathbb{Q}$ (in particular those dividing n) and enough others that the ring $\mathfrak{o}_S$ of S-integers is a principal ideal domain. We recall that $\mathfrak{o}_S$ is the set of elements $\alpha \in F$ such that $|\alpha_v|_v \leqslant 1$ for all places v of F not in S. Such an S always exists, since we may take the union of the set of archimedean places, the set of primes ramified over $\mathbb{Q}$, and those dividing a fixed set of generators of the ideal class group of the ring of integers in F. We embed $\mathfrak{o}_S$ in $F_S = \prod_{v \in S} F_v$ along the diagonal. It is discrete and cocompact.

We will denote by $\left(\frac{c}{d}\right)$ the n-th power residue symbol, defined when c and d are coprime elements of $\mathfrak{o}_S$ and $\gcd(n,d)=1$. It depends only on c modulo d, and satisfies the reciprocity law

$$\left(\frac{c}{d}\right)=(d,c)\left(\frac{d}{c}\right),$$

where $(c,d)\in\mu_n$ denotes the Hilbert symbol, defined for $c,d\in F_S^\times$. We also choose a nontrivial additive character ψ of F_S such that $\psi(x\mathfrak{o}_S)=1$ if and only if $x\in\mathfrak{o}_S$. (See Brubaker and Bump [2], Lemma 1.) If t is any positive integer and $a,c\in\mathfrak{o}_S$ we define the Gauss sum

$$g_t(a,c)=\sum_{d \bmod c}\left(\frac{d}{c}\right)^t\psi\left(\frac{ad}{c}\right). \tag{1}$$

The properties of the Hilbert and power residue symbols and Gauss sums will be reviewed later in Section 2.

We are now ready to define the Weyl group multiple Dirichlet series. Let $\varDelta$ be the set of simple positive roots in $\varPhi$. There will be one complex parameter s_i for each $\alpha_i\in\varDelta$. We also denote $\boldsymbol{s}=(s_1,\cdots,s_r)$; as we will explain in Section 3, the domain space $\mathbb{C}^r$ of $\boldsymbol{s}$ can be naturally regarded as the complexified dual of the real vector space $\mathbb{R}^r$ that contains $\varPhi$.

The Weyl group multiple Dirichlet series has the form

$$Z_\varPsi(\boldsymbol{s})=Z_\varPsi(s_1,\cdots,s_r)=\\ \sum_{0\neq C_1\in\mathfrak{o}_S^\times\backslash\mathfrak{o}_S}\cdots\sum_{0\neq C_r\in\mathfrak{o}_S^\times\backslash\mathfrak{o}_S} H(C_1,\cdots,C_r)\,\varPsi(C_1,\cdots,C_r)\,\mathbb{N}C_1^{-2s_1}\mathbb{N}C_2^{-2s_2}\cdots\mathbb{N}C_r^{-2s_r}, \tag{2}$$

where the functions H and $\varPsi$ will be described below. The summand will be unchanged if C_i is changed by a unit, so $Z_\varPsi$ may be regarded as a sum over r-tuples of ideals of $\mathfrak{o}_S$. (See Lemma 2.9.)

The function $H(C_1,\cdots,C_r)$ is defined if $C_1,\cdots,C_r$ are nonzero elements of $\mathfrak{o}_S$. To specify it, we require first that it obeys the following "twisted" multiplicativity. If

$$\gcd(C_1\cdots C_r, C_1'\cdots C_r')=1, \tag{3}$$

then

$$\frac{H(C_1C_1',\cdots,C_rC_r')}{H(C_1,\cdots,C_r)\,H(C_1',\cdots,C_r')}=\\ \prod_{i=1}^r\left(\frac{C_i}{C_i'}\right)^{\|\alpha_i\|^2}\left(\frac{C_i'}{C_i}\right)^{\|\alpha_i\|^2}\prod_{i<j}\left(\frac{C_i}{C_j'}\right)^{2\langle\alpha_i,\alpha_j\rangle}\left(\frac{C_i'}{C_j}\right)^{2\langle\alpha_i,\alpha_j\rangle}. \tag{4}$$

Here we have chosen a Weyl group-invariant inner product $\langle\alpha,\beta\rangle$ on $\mathbb{R}^r$ that is normalized so that if $\alpha,\beta\in\varPhi$ then $\|\alpha\|^2=\langle\alpha,\alpha\rangle$ and $2\langle\alpha,\beta\rangle$ are integers. In particular, if $\varPhi$ is irreducible the inner product can be chosen so that all short roots have length 1. For example, if $\varPhi$ is of type A_2 and the roots are normalized to have length 1 then

$$H(C_1C_1',C_2C_2')=H(C_1,C_2)H(C_1',C_2')\left(\frac{C_1}{C_1'}\right)\left(\frac{C_1'}{C_1}\right)\left(\frac{C_2}{C_2'}\right)\left(\frac{C_2'}{C_2}\right)\left(\frac{C_1}{C_2'}\right)^{-1}\left(\frac{C_1'}{C_2}\right)^{-1}.$$

The "twistedness" of the multiplicativity (4) means that Z_Ψ is not an Euler product. On the other hand, this multiplicativity reduces specification of the coefficients H to the case where the C_i are all powers of the same prime p. We describe these now.

For each prime p, the coefficient $H(p^{k_1}, \cdots, p^{k_r}) = 0$ except for a finite number of particular values of $(k_1, \cdots, k_r)$; these values are in one-to-one correspondence with the elements of the Weyl group W, which acts on $\mathbb{R}^r$. To describe these nonzero values, let $\rho \in \mathbb{R}^r$ be half the sum of the positive roots. Given $w \in W$, the parametrized value of $(k_1, \cdots, k_r)$ is determined by the condition that

$$\rho - w\rho = \sum_{\alpha_i \in \Delta} k_i \alpha_i .$$

Define a function d on the set Φ^+ of positive roots, taking values in the positive integers, by

$$d(\alpha) = \frac{2\langle \rho, \alpha \rangle}{\langle \alpha, \alpha \rangle} . \tag{5}$$

Let Φ_w be the set of all positive roots α such that $w(\alpha)$ is a negative root. The cardinality of Φ_w is equal to the length $l(w)$ of w in the Weyl group. Then

$$H(p^{k_1}, \ldots, p^{k_r}) = \prod_{\alpha \in \Phi_w} g_{\|\alpha\|^2}(p^{d(\alpha)-1}, p^{d(\alpha)}) . \tag{6}$$

If $w = 1$ then $\Phi_w = \varnothing$, so $H(1, \cdots, 1) = 1$. By Proposition 2.7 (iv) we can also write

$$H(p^{k_1}, \ldots, p^{k_r}) = \prod_{\alpha \in \Phi_w} \mathbb{N}p^{d(\alpha)-1} \cdot g_{d(\alpha)\|a\|^2}(1, p) .$$

The function Ψ is drawn from a finite-dimensional vector space of functions on $(F_S^\times)^r$ whose definition is given later in (12). If $\varepsilon_1, \cdots, \varepsilon_r \in \mathfrak{o}_S^\times$, then (4) shows how $H(C_1, \cdots, C_r)$ changes if we replace C_i by $\varepsilon_i C_i$, and (12) implies a compensating change in $\Psi(C_1, \cdots, C_r)$, so that Z_Ψ is well-defined.

Remark 1.1. The definitions (12) and (4) depend on the choice of ordering of the simple roots α_i for $i = 1, \ldots, r$. In all that follows, we will assume that such a choice is fixed. If one took a different ordering of the roots, then the residue symbols and Hilbert symbols appearing in all subsequent definitions would be slightly altered, but the space of multiple Dirichlet series that we define would be unchanged, since there would be compensating changes to both H and the space of Ψ in (2).

At this point, we have completely defined the Weyl group multiple Dirichlet series that are the subject of this paper. Our main result is as follows.

Main Theorem. *The function $Z_\Psi(s_1, \cdots, s_r)$, originally convergent when $\Re(s_i)$ are sufficiently large, has meromorphic continuation to all $\mathbb{C}^r$ and satisfies functional equations with respect to a group of transformations of $\mathbb{C}^r$ isomorphic to the Weyl group W.*

A more precise version of this theorem may be found in Theorem 5.9. The action of W on $\mathbb{C}^r$ is described in Section 3.

This paper is one of several recent ones on this subject. The paper of Brubaker, Bump, Chinta, Friedberg and Hoffstein [3] gives specific examples and an expanded discussion; it can be regarded as an introduction to this paper. The paper of Brubaker and Bump [2] treats the case $\Phi = A_1$ (in Cartan's classification) and contains a specific foundational result that we will require, while [3] discusses A_2 and C_2. The paper of Brubaker, Bump, Friedberg and Hoffstein [4] returns to A_2, introducing instability through twisting; it also marks progress towards the removal of the stability assumption, giving a conjectural description of the Weyl group multiple Dirichlet series when $\Phi = A_r$ that is valid for all n. This description expresses the coefficients of the multiple Dirichlet series as a sum of products of Gauss sums indexed by Gelfand-Tsetlin patterns, and has some common characteristics with the formula of Casselman and Shalika [7] for spherical Whittaker functions. Using a different approach, the earlier paper of Chinta [8] applies the case $n = 2$ and $\Phi = A_5$ to a problem in analytic number theory; this example is outside the stable range. What is specific to the current paper is the complete proof of the analytic continuation and functional equations of the Weyl group multiple Dirichlet series for general Φ in the stable range.

We believe that these multiple Dirichlet series can be understood in the context of automorphic forms. More precisely, let G be a simply connected algebraic group over F whose root system is the *dual* of the root system Φ; in other words, Φ is the root system of the L-group ${}^LG$.

Conjecture 1.2. The Weyl group multiple Dirichlet series $Z_\Psi(s_1, \cdots, s_r)$ is a Whittaker coefficient of an Eisenstein series on an n-fold metaplectic cover $\tilde{G}$ of G.

The evidence for this conjecture includes the following points.

- The normalizing factors and groups of functional equations for Z_Ψ are the same as for the Eisenstein series on $\tilde{G}$. The normalizing factors can be computed by the formula of Gindikin and Karpelevich; this computation is similar to the normalization of the intertwining integrals on the metaplectic group in Kazhdan and Patterson [12], Proposition I.2.4. The normalizing factors are in bijection with the positive roots, and their form depends on $n(\alpha)$ as defined in Section 3, precisely as in (24).
- It is clear that the Whittaker coefficients of metaplectic Eisenstein series are multiple Dirichlet series whose coefficients involve Gauss sums.
- We compute the Whittaker coefficients of Eisenstein series on the metaplectic covers of GL_3 in [4], and we find that they agree with the Weyl group multiple Dirichlet series of type A_2 as defined in this paper. We refer to [4] for further discussion of the conjecture.

The method by which we prove our main theorem, based on Hartogs' theorem, is similar to the method by which Jacquet proved the functional equations of Whittaker functions in [11]; the same method underlies the reduction of the functional equations of Eisenstein series to the rank one case, and goes back to Selberg [16].

It should be noted that while Conjecture 1.2 would imply a version of Theorem 5.9, the precise statement of Theorem 5.9 would be difficult to obtain by that method. Indeed, we rely on the precise results in Brubaker and Bump [2] that are obtained by proving local functional equations for Dirichlet series arising from test data with small support in the big Bruhat cell of SL_2. Adapting this argument to an arbitrary G would be harder than the argument we give here. Thus we believe that the best way to study the analytic properties of the Weyl group multiple Dirichlet series is that of this paper, though the connection with Eisenstein series is important for philosophical reasons.

The Weyl group multiple Dirichlet series unify and extend many examples that have previously been treated on a case-by-case basis. This is particularly true when one considers residues of these multiple Dirichlet series. Some of these instances are surveyed in the final Section of [3].

To provide the reader with a glimpse of the possibilities that appear when one considers the residues of Weyl group multiple Dirichlet series, we discuss one particular example in detail – a connection with the still-mysterious Whittaker-Fourier coefficients of the generalized theta series of Kubota [13] and Kazhdan and Patterson [12]. These automorphic forms, which are residues of Eisenstein series on the n-fold metaplectic cover of GL_r, are intimately connected with the reciprocity laws. Their Fourier-Whittaker coefficients are incompletely known except when $r = n$ or $n-1$, in which case they are products of Gauss sums. Yet there is strong evidence that unproved relations exist between these coefficients, such as the one conjectured by Patterson [15], and those conjectured by Bump and Hoffstein [6].

The Fourier coefficients $\tau(c)$ of the generalized theta series $\theta^{(4)}$ on the 4-fold cover of GL(2) were studied by Suzuki [18], who was able to evaluate only some of these coefficients, for reasons which eventually were understood as relating to the nonuniqueness of Whittaker models. Still Patterson [15] was able to make a conjecture about the nature of these coefficients. Despite partial proofs by Suzuki in [19] and [20], the Patterson conjecture remains unproved. We will explain that the conjecture amounts to a relationship between specializations of two different Weyl group multiple Dirichlet series.

Since different root systems Φ and different values of n will appear in the following discussion, we will denote $Z_\Psi(\boldsymbol{s})$ by $Z(\boldsymbol{s};\Phi^{(n)})$; we suppress the Ψ since it plays no important role in this heuristic discussion. The function $Z(s_1,s_2,s_3;A_3^{(n)})$ is in the stable range if $n \geqslant 3$. It is a function of three complex variables, s_1, s_2 and s_3, and taking the residue in s_3 yields a function of two complex variables. One can compute this residue using Proposition 5.6; that proposition expresses Z in terms of Kubota Dirichlet series $\mathcal{D}$ introduced below in Section 2.4. The coefficients τ appear as a consequence of taking the residue of the Kubota Dirichlet series. One may then take a second residue in s_1 obtaining a Dirichlet series $\Upsilon^{(n)}(s)$ in one variable $s = s_2$. The Dirichlet series $\Upsilon^{(n)}(s)$ is not multiplicative, but for the purpose of discussion and comparison with Bump and Hoffstein [6] we give its p-part, where p is a prime of F. Using results of this paper this can be computed to be

$$1 + g(1,p)\mathbb{N}p^{-\frac{1}{n}-2s_2}\tau(p)^2 + 2g(1,p)g(p,p^2)\mathbb{N}p^{-1-\frac{2}{n}-4s_2}\tau(p^2) +$$
$$g(1,p)^2 g(p^2,p^3)\mathbb{N}p^{-2-\frac{3}{n}-6s_2}\tau(p)^2 + g(1,p)g(p,p^2)^2 g(p^2,p^3)\mathbb{N}p^{-4-\frac{4}{n}-8s_2}.$$

The Patterson conjecture states that when $n = 4$, we have $\tau(p)^2 = 2\,\overline{g(1,p)}\,\mathbb{N}p^{-1/2}$.

The Dirichlet series $\Upsilon^{(4)}(s)$ was constructed by Bump and Hoffstein [6] as a Whittaker coefficient of a metaplectic Eisenstein series on the 4-fold metaplectic cover of GL_4, induced from two copies of $\theta^{(4)}$. We note that this equivalence of $\Upsilon(s)$ with their Dirichlet series can be checked directly, but it would follow from Conjecture 1.2, and so it is evidence for Conjecture 1.2.

Bump and Hoffstein [6] described a relationship that can be expressed in the equation

$$\Upsilon^{(4)}(s) = \zeta_F(8s-2)\zeta_F(8s-3)\, Z\left(s+\frac{1}{8}, s+\frac{1}{8}; A_2^{(2)}\right).$$

Much evidence supports this guess, which is equivalent to the Patterson conjecture.

Let us make a remark on the poles of these Dirichlet series. It follows from Theorem 5.9 that $\Upsilon^{(4)}$ has the following poles:

s	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{5}{8}$
multiplicity	1	2	2	2	1

Dividing by $\zeta_F(8s-2)\zeta_F(8s-3)$this gives a Dirichlet series with the following poles:

s	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{5}{8}$
multiplicity	1	1	1	1

(7)

Also by Theorem 5.9 the function

$$Z(s_1, s_2; A_2^{(2)})$$

has poles along the lines $s_1, s_2, s_1+s_2-\frac{1}{2} = \frac{1}{4}, \frac{3}{4}$. This would suggest that $Z(s+\frac{1}{8}, s+\frac{1}{8}; A_2^{(2)})$ should have the poles in (7), but that the poles at $\frac{1}{8}$ and $\frac{5}{8}$ should be double. That this doubling of the poles is illusory can be seen from Conjecture 1.2 as follows. The Taylor expansion of $Z(s_1, s_2; A_2^{(2)})$ near $s_1 = s_2 = \frac{3}{4}$ has the form $F(s_1, s_2)(s_1-\frac{3}{4})^{-1}(s_2-\frac{3}{4})^{-1}$, and by Conjecture 1.2 the function $F(s_1, s_2)$ should be a Whittaker coefficient of the theta function on the double cover of GL_3. Yet Kazhdan and Patterson [12] have shown that this theta function has no Whittaker model, so $F(\frac{3}{4}, \frac{3}{4}) = 0$. This causes the disappearance of one of the poles of

$$Z\left(s+\frac{1}{8}, s+\frac{1}{8}; A_2^{(2)}\right) = F\left(s+\frac{1}{8}, s+\frac{1}{8}\right)\left(s-\frac{5}{8}\right)^{-2},$$

consistent with the poles of $\zeta_F(8s-2)^{-1}\zeta_F(8s-3)^{-1}\Upsilon^{(4)}(s)$ described above.

This work was supported by NSF Grants FRG DMS-0354662 and FRG DMS-0353964. We would like to thank the referee for many useful comments. We would also like to thank Gautam Chinta and Jeffrey Hoffstein for their sustained interest in this project.

2. Preliminaries

In this section we collect some foundational material, and describe our basic setup.

Let $n > 1$ be an integer and let F, S, $\mathfrak{o}_S$, ψ and F_S be as in the introduction. We will partition $S = S_\infty \cup S_{\text{fin}}$, where S_∞ is the set of archimedean places, and S_{fin} are the remaining nonarchimedean ones. We will denote

$$F_S = \prod_{v \in S} F_v \quad \text{with} \quad F_\infty = \prod_{v \in S_\infty} F_v, \qquad F_{\text{fin}} = \prod_{v \in S_{\text{fin}}} F_v,$$

where S_∞ is the set of archimedean places in S, and S_{fin} is the set of nonarchimedean ones.

2.1. Hilbert symbol

Remark 2.1. In what follows, we will maintain an ambiguity whether the group μ_n is a subgroup of $F^\times$ or of $\mathbb{C}^\times$. To be correct, μ_n should be regarded as a subgroup of $F^\times$, and there will be fixed an embedding $j : \mu_n \longrightarrow \mathbb{C}^\times$; we suppress j from the notation.

If v is a place of F, the v-Hilbert symbol $(\ ,\ )_v$ is a map $F_v^\times \times F_v^\times \longrightarrow \mu_n$. We will take the symbol to be the *inverse* of the symbol defined by Neukirch [14].

Proposition 2.2. *(i) The Hilbert symbol is a skew-symmetric bilinear pairing on* $F_v^\times$. *That is,*

$$\begin{aligned} (aa', b)_v &= (a, b)_v (a', b)_v, \\ (a, bb')_v &= (a, b)_v (a, b')_v, \\ (a, b)_v &= (b, a)_v^{-1}. \end{aligned}$$

(ii) We have $(a, b)_v = 1$ *if and only if* a *is a norm from* $F_v(b^{1/n})$.
(iii) We have $(a, -a)_v = 1$, *and if* $a \neq 1$ *we have* $(a, 1 - a)_v = 1$.
(iv) If v *is complex, then* $(a, b)_v = 1$ *for all* $a, b \in F_v^\times$.
(v) If v *is nonarchimedean and* n *is prime to the residue characteristic, then* $(a, b)_v = 1$ *when* a, b *are both elements of the group* $\mathfrak{o}_v^\times$ *of units of the ring* $\mathfrak{o}_v$ *of integers in* F_v.

Proof. See Neukirch [14], Chapter V Section 3, particularly Proposition 3.2 on p. 334. □

Theorem 2.3 (Hilbert). *If* $a, b \in F^\times$, *then*

$$\prod_v (a, b)_v = 1. \tag{8}$$

This is the *Hilbert reciprocity law.*

Proof. See Neukirch [14] Chapter VI, Theorem 8.1 on p. 414. □

Let $F_S = \prod_{v \in S} F_v$. Define a pairing $(\ ,\ ) : F_S^\times \times F_S^\times \longrightarrow \mu_n$ by

$$(a, b) = \prod_{v \in S} (a, b)_v.$$

If $a, b \in \mathfrak{o}_S$ then we may also write

$$(a, b) = \prod_{v \notin S} (a, b)_v^{-1}$$

by the Hilbert reciprocity law (8). We will refer to this pairing $(\ ,\ )$ as the *Hilbert Symbol*.

Let Ω be a subgroup of $F_S^\times$. We say that Ω is *isotropic* if $(\varepsilon, \delta) = 1$ for $\varepsilon, \delta \in \Omega$. For example consider

$$\Omega_0 = \mathfrak{o}_S^\times F_S^{\times, n},$$

where $F_S^{\times, n}$ is the subgroup of n-th powers in $F_S^\times$.

Lemma 2.4. *The group Ω_0 is isotropic.*

Proof. See [2], Lemma 2. Note that this makes use of our assumption that -1 is an n-th power.

Let $\Omega \supset \Omega_0$ be an isotropic subgroup; there are advantages to taking Ω to be maximal isotropic, but we will not need this assumption. If t is a positive integer, let $\mathcal{M}_t(\Omega)$ be the vector space of functions $\Psi : F_{\text{fin}}^\times \longrightarrow \mathbb{C}$ that satisfy

$$\Psi(\varepsilon c) = (c, \varepsilon)^{-t} \Psi(c), \tag{9}$$

when $\varepsilon \in \Omega$. We denote $\mathcal{M}_1(\Omega)$ by $\mathcal{M}(\Omega)$. Note that if ε is sufficiently close to the identity in $F_{\text{fin}}^\times$ it is an n-th power at every place in S_{fin}, so such a function is locally constant. It is easy to see that the dimension of $\mathcal{M}(\Omega)$ is $[F_S^\times : \Omega] < \infty$.

2.2. Power residue symbol

If $a \in \mathfrak{o}_S$ and $\mathfrak{b}$ is an ideal of $\mathfrak{o}_S$ then the *power residue symbol* $\left(\frac{a}{\mathfrak{b}}\right)$ is defined as follows. If a is not coprime to $\mathfrak{b}$ then it is defined to be zero. If $\mathfrak{b} = \mathfrak{p}$ is prime and a is coprime to $\mathfrak{p}$, then $\left(\frac{a}{\mathfrak{p}}\right)$ is defined to be the unique element of μ_n such that

$$\left(\frac{a}{\mathfrak{p}}\right) \equiv a^{(\mathbb{N}\mathfrak{p}-1)/n} \mod \mathfrak{p}.$$

Note that $(\mathbb{N}\mathfrak{p}-1)/n$ is an integer since we are assuming that $\mu_n \subset F$. Finally, the definition of $\left(\frac{a}{\mathfrak{b}}\right)$ is extended to all $\mathfrak{b}$ by multiplicativity, that is, by the condition

$$\left(\frac{a}{\mathfrak{b}\mathfrak{c}}\right) = \left(\frac{a}{\mathfrak{b}}\right)\left(\frac{a}{\mathfrak{c}}\right).$$

If $a, b \in \mathfrak{o}_S$ we will also denote by $\left(\frac{a}{b}\right) = \left(\frac{a}{b\mathfrak{o}_S}\right)$ where $b\mathfrak{o}_S$ is the principal ideal generated by b.

Proposition 2.5. (Gauss, Eisenstein, Kummer, Hilbert) *If a and b are coprime then*

$$\left(\frac{a}{b}\right) = (b, a) \left(\frac{b}{a}\right). \tag{10}$$

Proof. See Neukirch [14], Theorem 8.3 of Chapter 6. □

Lemma 2.6. *If $\varepsilon \in \mathfrak{o}_S^\times$ and $0 \neq c \in \mathfrak{o}_S$ then*

$$\left(\frac{\varepsilon}{c}\right) = (c, \varepsilon). \tag{11}$$

Proof. This follows from (10) since ε is a unit, so $\left(\frac{c}{\varepsilon}\right) = 1$. □

If $a, c \in \mathfrak{o}_S$ and $c \neq 0$, and if t is a positive integer, define the Gauss sum $g_t(a, c)$ by (1). We will also denote $g_1(a, c) = g(a, c)$.

Proposition 2.7. *The Gauss sum has the following properties.*
(i) We have

$$g(m, cc') = \left(\frac{c}{c'}\right) \left(\frac{c'}{c}\right) g(m, c)\, g(m, c'), \qquad \text{if } c, c' \text{ are coprime.}$$

(ii) We have

$$g(am, c) = \left(\frac{a}{c}\right)^{-1} g(m, c) \qquad \text{if } a, c \text{ are coprime.}$$

(iii) If $\varepsilon \in \mathfrak{o}_S^\times$ we have

$$g(a, \varepsilon c) = (c, \varepsilon) g(a, c).$$

(iv) If $c \in \mathfrak{o}_S$ let $\phi(c)$ denote the cardinality of $(\mathfrak{o}_S / c\mathfrak{o}_S)^\times$. If p is prime, then

$$g_t(p^k, p^l) = \begin{cases} \mathbb{N}(p)^k\, g_{tl}(1, p) & \text{if } l = k + 1; \\ \phi(p^l) & \text{if } n | tl \text{ and } k \geqslant l; \\ 0 & \text{otherwise.} \end{cases}$$

(v) If a and p are coprime, then

$$|g(a, p)| = \sqrt{\mathbb{N}p}.$$

Proof. These properties of the Gauss sums are well known. See, for example, Ireland and Rosen [10]. Note that (iii) follows from (i) and (11). □

2.3. The vector space $\mathcal{M}(\Omega^r)$

Let $\mathcal{M}(\Omega^r)$ be the space of functions on $\Psi : (F_S^\times)^r \longrightarrow \mathbb{C}$ that satisfy

$$\Psi(\varepsilon_1 C_1, \cdots, \varepsilon_r C_r) = \prod_{i=1}^{r} (\varepsilon_i, C_i)^{\|\alpha_i\|^2} \left\{ \prod_{i<j} (\varepsilon_i, C_j)^{2\langle \alpha_i, \alpha_j \rangle} \right\} \Psi(C_1, \cdots, C_r) \tag{12}$$

when $\varepsilon_1, \cdots, \varepsilon_r \in \Omega$ and $C_i \in F_{\text{fin}}^\times$. This generalizes the definition of $\mathcal{M}(\Omega)$ in (9).

Lemma 2.8. *The dimension of* $\mathcal{M}(\Omega^r)$ *is* $|\Omega \backslash (F_{\text{fin}}^\times)|^r < \infty$.

Proof. Using the fact that Ω is isotropic, it is easy to see that the value Ψ may be specified arbitrarily on a set of representatives of $\Omega^r \backslash (F_{\text{fin}}^\times)^r$. □

Lemma 2.9. *The multiple Dirichlet series* Z_Ψ *is well defined.*

Proof. It must be checked that $H(C_1, \cdots, C_r)\Psi(C_1, \cdots, C_r)$ is unchanged if we replace C_i by $\varepsilon_i C_i$ for $\varepsilon_i \in \mathfrak{o}_S^\times$. In (4) we may take $C_i' = \varepsilon_i$. The statement follows from (11) and (12). □

2.4. Kubota Dirichlet series

If $\Psi \in \mathcal{M}(\Omega)$, the space of functions defined in (9), let

$$\mathcal{D}_t(s, \Psi, a) = \sum_{0 \neq c \in \mathfrak{o}_S / \mathfrak{o}_S^\times} g_t(a, c)\, \Psi(c)\, \mathbb{N}(c)^{-2s} \quad .$$

We will also denote $\mathcal{D}_1(s, \Psi, a) = \mathcal{D}(s, \Psi, a)$. Here $\mathbb{N}(c) = |c|$ is the order of $\mathfrak{o}_S / c\mathfrak{o}_S$. The term $g(a, c)\, \Psi(c)\, \mathbb{N}(c)^{-2s}$ is independent of the choice of representative c modulo S-units by (9) and Proposition 2.7. It follows easily from Proposition 2.7 that the series is convergent if $\Re(s) > \frac{3}{4}$.

Let

$$\boldsymbol{G}_n(s) = (2\pi)^{-2(n-1)s} n^{2ns} \prod_{j=1}^{n-1} \Gamma\left(2s - 1 + \frac{j}{n}\right). \tag{13}$$

In view of the multiplication formula for the Gamma function, we may also write

$$\boldsymbol{G}_n(s) = (2\pi)^{-(n-1)(2s-1)} \frac{\Gamma(n(2s-1))}{\Gamma(2s-1)}.$$

Let

$$\mathcal{D}^*(s, \Psi, a) = \mathbf{G}_n(s)^{\frac{1}{2}[F:\mathbb{Q}]} \zeta_F(2ns - n + 1)\, \mathcal{D}(s, \Psi, a)$$

where we note that $\frac{1}{2}[F : \mathbb{Q}]$ is the number of archimedean places of the totally complex field F and ζ_F is the Dedekind zeta function of F.

If $v \in S_{\text{fin}}$ let q_v denote the cardinality of the residue class field $\mathfrak{o}_v/\mathfrak{p}_v$, where $\mathfrak{o}_v$ is the local ring in F_v and $\mathfrak{p}_v$ is its prime ideal. By an *S-Dirichlet polynomial* we mean a polynomial in q_v^{-s} as v runs through the finite number of places in S_{fin}.

If $\Psi \in \mathcal{M}(\Omega)$ and $\eta \in F_S^\times$ denote

$$\tilde{\Psi}_\eta(c) = (\eta, c)\,\Psi(c^{-1}\eta^{-1}). \tag{14}$$

It is easy to check that $\tilde{\Psi}_\eta \in \mathcal{M}(\Omega)$ and that it depends only on the class of η in $F_S^\times/F_S^{\times,n}$.

The following result will be fundamental in our proofs.

Theorem 2.10. (Brubaker and Bump [2]) *Let $\Psi \in \mathcal{M}(\Omega)$, and let $a \in \mathfrak{o}_S$. Then $\mathcal{D}^*(s, \Psi, a)$ has meromorphic continuation to all s, analytic except possibly at $s = \frac{1}{2} \pm \frac{1}{2n}$, where it might have simple poles. There exist S-Dirichlet polynomials $P_\eta(s)$ that depend only on the image of η in $F_S^\times/F_S^{\times,n}$ such that*

$$\mathcal{D}^*(s, \Psi, a) = \mathbb{N}(a)^{1-2s} \sum_{\eta \in F_S^\times/F_S^{\times,n}} P_{a\eta}(s)\mathcal{D}^*(1-s, \tilde{\Psi}_\eta, a). \tag{15}$$

This result, based on ideas of Kubota [13], relies on the theory of Eisenstein series.

2.5. Root systems

Background information on root systems and Weyl groups may be found in Bump [5], Chapter 21, or Bourbaki [1], as well as many other sources. A *root system of rank r* is a finite subset Φ of nonzero vectors in the Euclidean space $\mathbb{R}^r$ that span $\mathbb{R}^r$ as a vector space satisfying the following properties. If $\alpha \in \Phi$, let $\sigma_\alpha : \mathbb{R}^r \longrightarrow \mathbb{R}^r$ denote the reflection in the hyperplane through the origin perpendicular to the vector α. The definition of a root system requires that $\sigma_\alpha(\Phi) = \Phi$, and if $\alpha, \beta \in \Phi$, then $\beta - \sigma_\alpha(\beta)$ is an integer multiple of α. We will denote by $\langle\ ,\ \rangle$ the Euclidean inner product on $\mathbb{R}^r$, and by $\|\alpha\| = \sqrt{\langle \alpha, \alpha \rangle}$ the Euclidean norm. We make the following

Normalization Assumption: *We require that $\|\alpha\|^2 \in \mathbb{Z}$ and that $2\langle \alpha, \beta \rangle \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$.*

Let us observe that this can always be arranged. There is no loss of generality in assuming that Φ is irreducible, that is, that there is no way to write $\Phi = \Phi_1 \cup \Phi_2$ where Φ_1 and Φ_2 lie in orthogonal subspaces of $\mathbb{R}^r$. Assuming this, the normalization assumption will be satisfied if we adjust the inner product so that the short roots have length 1.

We have

$$\sigma_\alpha(\beta) = \beta - \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha, \tag{16}$$

so

$$\frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$$

when α and β are roots. Since $-\alpha = \sigma_\alpha(\alpha)$, these axioms imply that $-\alpha \in \Phi$. The root system is called *reduced* if α and 2α are not both in Φ, and it is called *irreducible* if it is not the union of two smaller root systems that span orthogonal subspaces of $\mathbb{R}^r$.

Let $\Phi \subset \mathbb{R}^r$ be a reduced root system of rank r. We partition Φ as usual into positive roots Φ^+ and negative roots Φ^-. Let $\Delta = \{\alpha_1, \cdots, \alpha_r\} \subset \Phi^+$ be the subset of *simple positive roots*. These are the positive roots that cannot be decomposed into sums of other positive roots. It is well-known that α_i are a linear basis of $\mathbb{R}^r$, and that if $\alpha \in F$, then $\alpha = \sum k_i \alpha_i$ where $k_i \in \mathbb{Z}$; moreover, k_i are all positive if $\alpha \in \Phi^+$, and all negative if $\alpha \in \Phi^-$. The σ_{α_i} are called *simple reflections*. We will sometimes denote $\sigma_i = \sigma_{\alpha_i}$.

Let W denote the Weyl group of Φ, which is the group of transformations of $\mathbb{R}^r$ generated by the reflection σ_α in the hyperplane through the origin perpendicular to the root α. The inner product $\langle\, , \,\rangle$ is W-invariant. It is well-known that W is generated by the σ_α for α in the set Δ of simple roots. To each element $w \in W$ let $\Phi_w \subset \Phi^+$ be

$$\Phi_w = \Phi^+ \cap w^{-1}\Phi^- = \{\alpha \,|\, \alpha \in \Phi^+, w(\alpha) \in \Phi^-\}.$$

We will denote

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha.$$

We also recall that the *length function* $l : W \longrightarrow \mathbb{Z}$ is defined by letting $l(w)$ be the smallest number of simple reflections into which $w \in W$ may be decomposed.

Lemma 2.11. *Let $w \in W$.*
(i) The cardinality of Φ_w is the length $l(w)$ of w.
(ii) We have

$$\rho - w(\rho) = \sum_{\alpha \in \Phi_w} (-w\alpha) = \sum_{\substack{\alpha \in \Phi^+ \\ w^{-1}\alpha \in \Phi^-}} \alpha \tag{17}$$

while

$$\rho - w^{-1}(\rho) = \sum_{\alpha \in \Phi_w} \alpha. \tag{18}$$

(iii) Express $\rho - w(\rho)$ as a linear combination of the simple roots:

$$\rho - w(\rho) = \sum_{i=1}^{r} k_i \alpha_i. \tag{19}$$

Then the k_i are nonnegative integers.
(iv) If $w, w' \in W$ such that $\rho - w(\rho) = \rho - w'(\rho)$ then $w = w'$.
(v) For every simple root α_i we have

$$\frac{2\,\langle \rho, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} = 1. \tag{20}$$

Proof. Part (i) follows from Proposition 21.2 of [5].

For (ii), note that $\rho - w(\rho)$ equals

$$\begin{aligned}
&\frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha - \frac{1}{2}\sum_{\alpha\in\Phi^+}w(\alpha)\\
&=\frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha - \frac{1}{2}\sum_{w^{-1}\alpha\in\Phi^+}\alpha = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha + \frac{1}{2}\sum_{w^{-1}\alpha\in\Phi^-}\alpha\\
&=\frac{1}{2}\sum_{\substack{\alpha\,\in\,\Phi^+\\ w^{-1}\alpha\,\in\,\Phi^+}}\alpha + \frac{1}{2}\sum_{\substack{\alpha\,\in\,\Phi^+\\ w^{-1}\alpha\,\in\,\Phi^-}}\alpha + \frac{1}{2}\sum_{\substack{w^{-1}\alpha\,\in\,\Phi^-\\ \alpha\,\in\,\Phi^+}}\alpha + \frac{1}{2}\sum_{\substack{w^{-1}\alpha\,\in\,\Phi^-\\ \alpha\,\in\,\Phi^-}}\alpha.
\end{aligned}$$

The first term and the last term cancel, but the second and third term are equal, so this equals the right-hand side of (17). On the other hand,

$$\sum_{\alpha\in\Phi_w}(-w\alpha) = \sum_{\substack{\alpha\,\in\,\Phi^+\\ w(\alpha)\,\in\,\Phi^-}}(-w\alpha) = \sum_{\substack{\alpha\,\in\,\Phi^-\\ w(\alpha)\,\in\,\Phi^+}}w(\alpha) = \sum_{\substack{w^{-1}(\alpha)\,\in\,\Phi^-\\ \alpha\,\in\,\Phi^+}}\alpha,$$

where we have replaced α first by $-\alpha$, then $w^{-1}(\alpha)$. Now (17) is proved. Replacing w by w^{-1} gives (18).

Since (17) shows that $\rho - w(\rho)$ is a sum of positive roots, it is a sum of simple positive roots, and (iii) is also clear.

For (iv), we have $w(\rho) = w'(\rho)$, so $w^{-1}w'$ fixes ρ. It follows that $\Phi_{w^{-1}w'}$ is the empty set, so $l(w^{-1}w') = 0$, that is, $w^{-1}w' = 1$.

For (v), when α_i is a simple root, σ_i changes α_i to $-\alpha_i$ and permutes the remaining positive roots. Thus $\sigma_i(\rho) = \rho - \alpha_i$. By (16) we obtain (20). □

As in the introduction, define the function d on Φ^+ by (5).

Lemma 2.12. *We have $d(\alpha) \in \mathbb{Z}^+$ for all $\alpha \in \Phi^+$, and $d(\alpha) = 1$ if α is a simple positive root.*

We recall that $\lambda \in \mathbb{R}^r$ is a *weight* if $2\left\langle \lambda,\alpha\right\rangle / \left\langle \alpha,\alpha\right\rangle \in \mathbb{Z}$ for all $\alpha\in\Phi$, and the weight is *dominant* if $2\left\langle \lambda,\alpha\right\rangle / \left\langle \alpha,\alpha\right\rangle \geqslant 0$ for all $\alpha\in\Phi^+$. It is well-known that ρ is a dominant weight; in fact it is the sum of the fundamental dominant weights ([5], Proposition 21.16). If ε_i are the fundamental dominant weights then $2\left\langle \varepsilon_i,\alpha_j\right\rangle / \left\langle \alpha_j,\alpha_j\right\rangle = \delta_{ij}$ (Kronecker delta) which implies that $d(\alpha_i) = 1$ for the simple roots α_i.

Lemma 2.13. *If Φ is reduced then given any simple root α_i, and $w \in W$ a fixed element of the Weyl group, then*

$$\sum_{\alpha\in\Phi_w}\frac{2\left\langle w\alpha,\alpha_i\right\rangle}{\left\langle \alpha_i,\alpha_i\right\rangle} = d - 1 \tag{21}$$

where $d = d(w^{-1}\alpha_i)$ is defined in (5).

Proof. By (18),

$$\sum_{\alpha\in\Phi_w}\frac{2\left\langle w\alpha,\alpha_i\right\rangle}{\left\langle\alpha_i,\alpha_i\right\rangle}=\frac{2\left\langle w(\rho-w^{-1}\rho),\alpha_i\right\rangle}{\left\langle\alpha_i,\alpha_i\right\rangle}=\frac{2\left\langle w\rho-\rho,\alpha_i\right\rangle}{\left\langle\alpha_i,\alpha_i\right\rangle}.$$

On the other hand, using the W-invariance of $\langle\ ,\ \rangle$ and (20) the right-hand side of (21) is

$$\frac{2\left\langle\rho,w^{-1}(\alpha_i)\right\rangle}{\left\langle\alpha_i,\alpha_i\right\rangle}-1=\frac{2\left\langle w\rho,\alpha_i\right\rangle}{\left\langle\alpha_i,\alpha_i\right\rangle}-\frac{2\left\langle\rho,\alpha_i\right\rangle}{\left\langle\alpha_i,\alpha_i\right\rangle},$$

and the statement follows. □

3. Stability, normalizing factors and Weyl group action

3.1. Stability

Let Φ be a reduced root system of rank r. We assume that $\langle\ ,\ \rangle$ satisfies the Normalization Assumption of Section 2.5. We will also make a certain *stability assumption* to the effect that n is sufficiently large, depending on Φ. Without this assumption, we expect that Z_Ψ can be defined and satisfies a similar group of functional equations, but that the definition of Z_Ψ will be more complicated. (See [4].)

Stability Assumption. *We will assume that* $n\geqslant d(\alpha)\cdot\gcd(n,\|\alpha\|^2)$ *for every positive root* α*.*

3.2. Normalizing factors

The multiple Dirichlet series must be normalized with gamma and zeta factors. Let

$$n(\alpha)=\frac{n}{\gcd(n,\|\alpha\|^2)}. \tag{22}$$

For example, if all roots have length 1 (so Φ is simply-laced), then we have $n(\alpha)=n$ for every $\alpha\in\Phi$. On the other hand, if Φ is not simply-laced but irreducible, and if $\langle\ ,\ \rangle$ is normalized so that the short roots have length 1, we have

$$n(\alpha)=\begin{cases} n \text{ if } \alpha \text{ is a short root,}\\ n \text{ if } \alpha \text{ is a long root and } \Phi\neq G_2, \text{ and } n \text{ is odd}\\ \frac{n}{2} \text{ if } \alpha \text{ is a long root and } \Phi\neq G_2, \text{ and } n \text{ is even}\\ n \text{ if } \alpha \text{ is a long root and } \Phi=G_2, \text{ and } 3\nmid n\\ \frac{n}{3} \text{ if } \alpha \text{ is a long root and } \Phi=G_2, \text{ and } 3|n. \end{cases}$$

If α is a positive root, write $\alpha=\sum k_i\alpha_i$ as before. Let ζ be the Dedekind zeta-function of F and

$$\zeta_\alpha(\boldsymbol{s})=\zeta\left(1+2n(\alpha)\sum_{i=1}^{r}k_i(s_i-\frac{1}{2})\right). \tag{23}$$

Also let

$$\mathbf{G}_\alpha(\boldsymbol{s}) = \boldsymbol{G}_{n(\alpha)}\left(\frac{1}{2} + \sum_{i=1}^{r} k_i \left(s_i - \frac{1}{2}\right)\right)$$

where $\boldsymbol{G}_n(s)$ is defined as in (13). We define the *normalized* multiple Dirichlet series by

$$Z^*_\Psi(\boldsymbol{s}) = \left[\prod_{\alpha\in\Phi^+} G_\alpha(\boldsymbol{s})\,\zeta_\alpha(\boldsymbol{s})\right] Z_\Psi(\boldsymbol{s}). \tag{24}$$

3.3. Weyl group action

We now describe the precise action of W on $(s_1, \cdots, s_r)$ that will be used to state the functional equations of Z_Ψ. We start with a pairing $\mu : \mathbb{R}^r \times \mathbb{C}^r \longrightarrow \mathbb{C}$ defined as follows. If $\alpha \in \mathbb{R}^r$ and $\boldsymbol{s} = (s_1, \cdots, s_r) \in \mathbb{C}^r$, write $\alpha = \sum k_i \alpha_i$, and let

$$\mu(\alpha, \boldsymbol{s}) = \sum k_i s_i. \tag{25}$$

If $w \in W$, we define a transformation $\mathbb{C}^r \longrightarrow \mathbb{C}^r$, also to be denoted w, by requiring that

$$\mu\left(w\alpha, w(\boldsymbol{s}) - \frac{1}{2}\rho^\vee\right) = \mu\left(\alpha, \boldsymbol{s} - \frac{1}{2}\rho^\vee\right), \tag{26}$$

where $\frac{1}{2}\rho^\vee = \left(\frac{1}{2}, \cdots, \frac{1}{2}\right)$ is the center of symmetry. (The explanation for this notation will be given at the end of the section.)

This transformation will appear also in the following guise. If p is a prime, let

$$M_p(\alpha; \boldsymbol{s}) = \prod_{i=1}^{r} \mathbb{N}p^{-2k_i s_i} = \mathbb{N}p^{-2\mu(\alpha,\boldsymbol{s})}.$$

Then (26) gives

$$M_p\left(w\alpha, w(\boldsymbol{s}) - \frac{1}{2}\rho^\vee\right) = M_p\left(\alpha, \boldsymbol{s} - \frac{1}{2}\rho^\vee\right). \tag{27}$$

In order to describe this action more explicitly in $\mathbf{s}$, it is enough to examine the simple reflections σ_i in the root α_i, which generate the group W. The effect of σ_i on the root system is given by the map

$$\sigma_i : \alpha \longmapsto \alpha - \frac{2\langle\alpha, \alpha_i\rangle}{\langle\alpha_i, \alpha_i\rangle}\alpha_i,$$

in terms of the standard Euclidean inner product $\langle\ ,\ \rangle$ on $\mathbb{R}^r$.

Proposition 3.1. *The action of σ_i on $\boldsymbol{s} = (s_1, \ldots, s_r)$ according to (26) is given by:*

$$s_j \longmapsto s_j - \frac{2\langle\alpha_j, \alpha_i\rangle}{\langle\alpha_i, \alpha_i\rangle}\left(s_i - \frac{1}{2}\right), \qquad j = 1, \cdots, r. \tag{28}$$

In particular, $s_i \longmapsto 1 - s_i$.

Note that $-\frac{2\langle \alpha_j,\alpha_i\rangle}{\langle \alpha_i,\alpha_i\rangle} > 0$ if $j \neq i$.

Proof. Take $\alpha = \alpha_j$. The right-hand side of (26) equals $s_j - \frac{1}{2}$. Denoting $\sigma_i \boldsymbol{s} = (s_1', \cdots, s_r')$, the left-hand side of (26) equals

$$\mu\left(\alpha_j - \frac{2\,\langle \alpha_i, \alpha_j\rangle}{\langle \alpha_i, \alpha_i\rangle}\alpha_i, \sigma_i \boldsymbol{s} - \rho^\vee\right) = (s_j' - \frac{1}{2}) - \frac{2\,\langle \alpha_i, \alpha_j\rangle}{\langle \alpha_i, \alpha_i\rangle}(s_i' - \frac{1}{2}).$$

So

$$s_j - \frac{1}{2} = (s_j' - \frac{1}{2}) - \frac{2\,\langle \alpha_i, \alpha_j\rangle}{\langle \alpha_i, \alpha_i\rangle}\left(s_i' - \frac{1}{2}\right). \tag{29}$$

This reduces to (28).

The Weyl group action has the following interpretation. We recall the notion of *root data*. See, for example, in Springer [17], Section 1. It is an alternative description of a root system. Let $V = \mathbb{R}^r$ be the vector space containing Φ; its dual $V^\vee$ contains a root system $\Phi^\vee$, whose elements are called *coroots*. There is a bijection $\alpha \longmapsto \alpha^\vee$ from $\Phi \longrightarrow \Phi^\vee$ that sends long roots to short roots and conversely. We write the dual pairing $V \times V^\vee \longrightarrow \mathbb{R}$ as $(x, y) \longmapsto \langle x, y\rangle$. (We have already used the notation $\langle\ ,\ \rangle$ for an inner product on V itself, but no confusion can result since we will refrain from identifying V and $V^\vee$.) We have $\langle \alpha, \alpha^\vee\rangle = 2$ and the simple reflection $s_\alpha : V \longrightarrow V$ corresponding to α is given by $s_\alpha(x) = x - \langle x, \alpha^\vee\rangle$, and similarly $s_{\alpha^\vee} : V^\vee \longrightarrow V^\vee$ is defined by $s_{\alpha^\vee}(x) = x - \langle \alpha, x\rangle$. It is assumed that $s_\alpha(\Phi) = \Phi$ and $s_{\alpha^\vee}(\Phi^\vee) = \Phi^\vee$.

We may identify $\mathbb{C} \otimes V^\vee$ with $\mathbb{C}^r$ in such a way that the pairing μ becomes $\langle\ ,\ \rangle$. If this identification is made, and if $\rho^\vee$ is half the sum of the positive coroots, then (20) applied to $\Phi^\vee$ implies that $\langle \alpha_i, \rho^\vee\rangle = 1$ for simple roots α_i, so in our notation $\rho^\vee = (1, \cdots, 1)$. Since W acts on V, it also acts on $V^\vee$, but we wish to modify this action so that the fixed point of the action is at $\frac{1}{2}\rho^\vee$, rather than the origin. This is the meaning of the shift in the definition (26).

4. Local computations

As before, we let Φ denote a general root system with r simple roots. The basic idea in our combinatorial proof is to rearrange our Dirichlet series $Z_\Psi(\mathbf{s})$ so that the inner sum is over C_i for some fixed index i. This allows us to prove that the resulting sum can be realized as a finite linear combination of Dirichlet polynomials in terms of primes dividing C_j $(j \neq i)$, multiplied by Kubota Dirichlet series of the form $D(s, \Psi, \alpha)$ introduced earlier. We will see the precise way in which α is related to the parameters C_j in the next section. Here we take a first step in this direction by proving a relationship between the functions $H(C_1, \cdots, C_r)$ on prime powers.

We recall that on prime powers, $H(p^{k_1}, \cdots, p^{k_r})$ has finitely many non-zero terms as the k_i range over the non-negative integers. They are in one-to-one correspondence with elements $w \in W$, the Weyl group. We say that $(k_1, \cdots, k_r) \in \mathbb{Z}^r$ is *associated* to $w \in W$ if (19) is satisfied; in this case, we write

$$(k_1, \cdots, k_r) = \mathrm{assoc}(w).$$

Proposition 4.1. *Let $w \in W$ be such that $l(\sigma_i w) = l(w) + 1$. Suppose that* $\operatorname{assoc}(w) = (k_1, \cdots, k_r) \in \mathbb{Z}^r$ *and* $\operatorname{assoc}(\sigma_i w) = (l_1, \cdots, l_r)$. *Let $d = d(w^{-1}\alpha_i)$ in the notation (5). Then*

$$l_j = \begin{cases} k_i + d & \text{if } j = i; \\ k_i & \text{if } j \neq i, \end{cases} \tag{30}$$

and

$$H(p^{l_1}, \dots, p^{l_r}) = g_{\|\alpha_i\|^2}(p^{d-1}, p^d)\, H(p^{k_1}, \dots, p^{k_r}). \tag{31}$$

Moreover

$$d - 1 = \sum_{j=1}^{r} \frac{-2\,\langle \alpha_j, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} k_j\ . \tag{32}$$

Proof. Propositions 21.2 and 21.10 of [5] show that the assumption that $l(\sigma_i w) = l(w) + 1$ implies that $w^{-1}(\alpha_i) \in \Phi^+$ is the unique root appearing in the difference of sets $\Phi_{\sigma_i w} - \Phi_w$. Now (31) follows from (6) since $\|w^{-1}\alpha_i\| = \|\alpha_i\|$. Moreover,

$$\begin{aligned}
\sum_{j=1}^{r} l_j \alpha_j &= \rho - \sigma_i w \rho = (\rho - w\rho) + w\rho - \sigma_i w \rho \\
[\text{by (16)}] &= (\rho - w\rho) + \frac{2\,\langle w\rho, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \alpha_i \\
[\text{by } W\text{-invariance of } \langle\ ,\ \rangle] &= (\rho - w\rho) + \frac{2\,\langle \rho, w^{-1}\alpha_i \rangle}{\langle w^{-1}\alpha_i, w^{-1}\alpha_i \rangle} \alpha_i \\
[\text{by definition of } d = d(w^{-1}\alpha_i)] &= \sum_{j=1}^{r} k_j \alpha_j + d\alpha_i,
\end{aligned}$$

whence (30). Finally, by (17) we have

$$\sum_{\alpha \in \Phi_w} (-w\alpha) = \rho - w\rho = \sum_{j=1}^{r} k_j \alpha_j .$$

Since by Lemma 2.13

$$d - 1 = \sum_{\alpha \in \Phi_w} \frac{2\,\langle w\alpha, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle}$$

we obtain (32). □

Proposition 4.2. *Let $d = d(w^{-1}(\alpha_i))$. For any $w \in W$, the monomial*

$$\mathbb{N}p^{(s_i - \frac{1}{2})(d-1)} \prod_{\alpha \in \Phi_w} M_p(-w\alpha; \boldsymbol{s})$$

is invariant under the transformation (28).

Proof. Reinterpreting this in terms of the action of W given by (26), we must show equivalently that

$$\frac{1}{2}(d-1)\alpha_i + \sum_{\substack{\alpha > 0 \\ w\alpha < 0}} (-w\alpha) = \frac{1}{2}(d-1)\alpha_i + \rho - w\rho \tag{33}$$

is orthogonal to α_i. We will show that the right-hand side of (33) is fixed by σ_i, i.e.,

$$\sigma_i\rho - \sigma_i w\rho = (d-1)\alpha_i + \rho - w\rho.$$

Since $\rho - \sigma_i\rho = \alpha_i$ we can write this more simply as $w\rho - \sigma_i w\rho = d\alpha_i$, and indeed

$$d\alpha_i = \frac{2\left\langle w^{-1}\alpha_i, \rho\right\rangle}{\langle \alpha_i, \alpha_i\rangle}\alpha_i = \frac{2\left\langle \alpha_i, w\rho\right\rangle}{\langle \alpha_i, \alpha_i\rangle}\alpha_i = w\rho - \sigma_i w\rho.$$

This concludes the proof of the Proposition. □

5. Global functional equations

Let $\sigma_i \in W$ be a fixed simple reflection about $\alpha_i \in \Phi$.

Definition 5.1. *We say that* $(C_1, \cdots, C_r)$ *in* $(\mathfrak{o}_S)^r$ *is* ***admissible*** *if, for each prime* p*, there exists a Weyl group element* $w_p \in W$ *such that*

$$(\mathrm{ord}_p(C_1), \cdots, \mathrm{ord}_p(C_r)) = \mathrm{assoc}(w_p).$$

For such $(C_1, \cdots, C_r)$*, we say that* C_i *is* ***i-reduced*** *if, for every* p*, we have* $l(\sigma_i w_p) = l(w_p) + 1$.

We note that if $C_1, \ldots, C_r$ are nonzero elements of $\mathfrak{o}_S$, then $(C_1, \ldots, C_r)$ is admissible if and only if $H(C_1, \cdots, C_r) \neq 0$. This is immediate from the definition of H.

Proposition 5.2. *Let* $C_1, \cdots, C_{i-1}, C_{i+1}, \cdots, C_r$ *be nonzero elements of* $\mathfrak{o}_S$*. If there exists a* C_i *such that* $(C_1, \cdots, C_r)$ *is admissible, then there exists a* C_i' *(modulo the action of* $\mathfrak{o}_S^\times$*) that is i-reduced. This* C_i' *divides* C_i *and is uniquely determined up to multiplication by a unit. Moreover, if*

$$(\mathrm{ord}_p(C_1), \cdots, \mathrm{ord}_p(C_i'), \cdots, \mathrm{ord}_p(C_r)) = (k_1, \cdots, k_r) = \mathrm{assoc}(w_p')$$

then either $\mathrm{ord}_p(C_i) = k_i$ *or* $\mathrm{ord}_p(C_i) = k_i + d$*, where* $d = d((w_p')^{-1}\alpha_i)$.

Proof. Let w_p be such that $(\mathrm{ord}_p(C_1), \cdots, \mathrm{ord}_p(C_r)) = \mathrm{assoc}(w_p)$. Let

$$w_p' = \begin{cases} w_p & \text{if } l(\sigma_i w_p) = l(w_p) + 1; \\ \sigma_i w_p & \text{if } l(\sigma_i w_p) = l(w_p) - 1. \end{cases}$$

In either case $l(\sigma_i w'_p) = l(w'_p) + 1$ and so there exists admissible $(C'_1, \cdots, C'_r)$ such that $(\mathrm{ord}_p(C'_1), \cdots, \mathrm{ord}_p(C'_r)) = \mathrm{assoc}(w'_p)$, and C'_i is i-reduced with respect to $C'_1, \cdots, C'_{i-1}$, $C'_{i+1}, \cdots, C'_r$. By Proposition 4.1, we have

$$\mathrm{assoc}(w_p) = \begin{cases} \mathrm{assoc}(w'_p) & \text{if } l(\sigma_i w_p) = l(w_p) + 1; \\ \mathrm{assoc}(w'_p) + (0, \cdots, d, \cdots, 0) & \text{if } l(\sigma_i w_p) = l(w_p) - 1, \end{cases}$$

where $d = d((w'_p)^{-1}\alpha_i)$. Thus C'_j differs from C_j by a unit if $j \neq i$, so we may as well take $C'_j = C_j$. Moreover, $\mathrm{ord}_p(C_i) \geqslant \mathrm{ord}_p(C'_i)$, so C'_i divides C_i. □

By Proposition 5.2,

$$Z_\Psi(s_1, \cdots, s_r) =$$

$$\sum_{\substack{0 \neq C_j \in \mathfrak{o}_S^\times \backslash \mathfrak{o}_S \\ 1 \leqslant j \leqslant r \\ C_1, \cdots, C_r \text{ admissible} \\ C_i \ i\text{-reduced}}} \mathbb{N}C_1^{-2s_1} \cdots \mathbb{N}C_r^{-2s_r} H(C_1, \cdots, C_r) \sum_{0 \neq D \in \mathfrak{o}_S^\times \backslash \mathfrak{o}_S} (D, C_i)^{\|\alpha_i\|^2}$$

$$\times \frac{H(C_1, C_2, \cdots, DC_i, \cdots, C_r)}{H(C_1, C_2, \cdots, C_i, \cdots, C_r)} \prod_{j>i} (D, C_j)^{2\langle \alpha_i, \alpha_j \rangle} \Psi_i^{C_1, \cdots, C_r}(D)\, \mathbb{N}D^{-2s_i}, \tag{34}$$

where we define

$$\Psi_i^{C_1, \ldots, C_r}(D) = \Psi(C_1, \ldots, C_i D, \ldots, C_r)(D, C_i)^{-\|\alpha_i\|^2} \prod_{j>i} (D, C_j)^{-2\langle \alpha_i, \alpha_j \rangle} \tag{35}$$

to emphasize the dependence on D for fixed parameters $C_1, \ldots, C_r$ in the inner sum. The choice of Hilbert symbols is explained by Lemma 5.4 below.

Remark 5.3. This is equivalent to summing over all C_j and noting that the exponential sum H is zero in almost all cases, but we want to emphasize this correspondence with the Weyl group. The non-zero terms in the inner sum will be those for which D corresponds to either w or $\sigma_i w$ at each prime p dividing C_j for some j, and hence the inner sum is finite as well.

From the previous section, quotients of H at prime powers, when grouped in pairs as above, are well-behaved. The following lemmas show that the inner sum is a Kubota Dirichlet series. First we need to explain some nuances related to the fact that Kubota Dirichlet series involving Gauss sums of differing degrees can occur.

Let $t = ||\alpha_i\|^2$; both t and i will be fixed in the following discussion. Here $t = 1, 2$ or 3 if Φ is irreducible and the short roots have length 1. If $\Psi \in \mathcal{M}_t(\Omega)$ then we may consider Dirichlet series of the form

$$\mathcal{D}_t^*(s, \Psi, a) = \mathbf{G}_m(s)^{\frac{1}{2}[F:\mathbb{Q}]}\, \zeta_F(2ms - m + 1)\, \mathcal{D}_t(s, \Psi, a), \tag{36}$$

where $m = n/\gcd(n, t)$ and

$$\mathcal{D}_t(s, \Psi, a) = \sum_{0 \neq c \in \mathfrak{o}_S / \mathfrak{o}_S^\times} g_t(a, c)\, \Psi(c)\, \mathbb{N}(c)^{-2s}.$$

These are Kubota Dirichlet series for Gauss sums relative to the symbol $\left(\frac{\cdot}{\cdot}\right)^t$. This is a power residue symbol of degree m; if $\gcd(n,t)=1$, this means that we still have functional equations for $\mathcal{D}_t$ implied by Theorem 2.10, but that the embedding $j:\mu_n\longrightarrow\mathbb{C}^\times$ in Remark 2.1 will be raised to the t-th power. In every case Theorem 2.10 is applicable with n replaced by m. Because $P_{\alpha\eta}$ in Theorem 2.10 will arise for different values of t (with n replaced by m) we will add t to the notation to write (15) in the form

$$\mathcal{D}_t^*(s,\Psi,a)=\mathbb{N}(a)^{1-2s}\sum_{\eta\in F_S^\times/F_S^{\times,n}}P_{a\eta}^t(s)\mathcal{D}_t^*(1-s,\tilde{\Psi}_\eta^t,a). \tag{37}$$

As in (14), if $\Psi\in\mathcal{M}_t(\Omega)$ and $\eta\in F_S^\times$ denote

$$\tilde{\Psi}_\eta^t(c)=(\eta,c)^t\,\Psi(c^{-1}\eta^{-1}). \tag{38}$$

Note that Theorem 2.10 would actually have us summing over $\eta\in F_S^\times/F_S^{\times,m}$ but there is no harm in summing over the larger set $\eta\in F_S^\times/F_S^{\times,n}$, and dividing the value of $P_{a\eta}^t(s)$ by a factor of $\gcd(n,t)$. What will prove important is that the factor $\mathbb{N}(a)^{1-2s}$ does not depend on t.

Lemma 5.4. *Let $C_1,\ldots,C_r$ be fixed nonzero elements of $\mathfrak{o}_S$. Then with the notation (35), the function $\Psi_i^{C_1,\ldots,C_r}\in\mathcal{M}_t(\Omega)$.*

Proof. We check the transformation property for $\Psi_i^{C_1,\ldots,C_r}(D)$ when D is multiplied by a unit ε, using the definition (12):

$$\begin{aligned}\Psi_i^{C_1,\ldots,C_r}(\varepsilon D)&=\Psi(C_1,\ldots,C_i\varepsilon D,\ldots,C_r)(\varepsilon D,C_i)^{-\|\alpha_i\|^2}\prod_{j>i}(\varepsilon D,C_j)^{-2\langle\alpha_i,\alpha_j\rangle}\\&=(\varepsilon,DC_i)^{\|\alpha_i\|^2}\left\{\prod_{j>i}(\varepsilon,C_j)^{2\langle\alpha_i,\alpha_j\rangle}\right\}\Psi(C_1,\ldots,C_iD,\ldots,C_r)\\&\quad\times(\varepsilon D,C_i)^{-\|\alpha_i\|^2}\prod_{j>i}(\varepsilon D,C_j)^{-2\langle\alpha_i,\alpha_j\rangle}\\&=(\varepsilon,D)^{\|\alpha_i\|^2}\Psi_i^{C_1,\ldots,C_r}(D).\end{aligned}$$

□

Lemma 5.5. *Fix an integer $i\in\{1,\ldots,r\}$. Given admissible $(C_1,\ldots,C_r)\in\mathfrak{o}_S^r$ with C_i i-reduced, then*

$$C_0=\prod_j C_j^{-2\langle\alpha_j,\alpha_i\rangle/\langle\alpha_i,\alpha_i\rangle} \tag{39}$$

is an $\mathfrak{o}_S$ integer and for every $D\in\mathfrak{o}_S$ we have

$$\frac{H(C_1,\ldots,DC_i,\ldots,C_r)}{H(C_1,\ldots,C_r)}(D,C_i)^{\|\alpha_i\|^2}\prod_{j>i}(D,C_j)^{2\langle\alpha_i,\alpha_j\rangle}=g_{\|\alpha_i\|^2}(C_0,D). \tag{40}$$

Moreover for each prime p *of* $\mathfrak{o}_S$ *we have*

$$\mathrm{ord}_p(C_0) = d(w_p^{-1}\alpha_i) - 1, \tag{41}$$

where w_p *is determined by the condition*

$$\mathrm{assoc}(w_p) = (\mathrm{ord}_p(C_1), \cdots, \mathrm{ord}_p(C_r)).$$

Proof. To prove (40), we first check that the multiplicativity of the left-hand side of (40) is the same as that of the Gauss sum. Let $h(D)$ denote the left-hand side. Let $\gcd(D, E) = 1$ and for $j = 1, \dots, r$, write $C_j = C'_j C''_j$ so that $\gcd(DC'_1 \cdots C'_r, EC''_1 \cdots C''_r) = 1$. To check the multiplicativity we consider $h(DE)h(D)^{-1}h(E)^{-1}$, which equals

$$\frac{H(C'_1C''_1, \dots, DEC'_iC''_i, \dots, C'_rC''_r)H(C'_1C''_1, \dots, C'_rC''_r)}{H(C'_1C''_1, \dots, DC'_iC''_i, \dots, C'_rC''_r)\, H(C'_1C''_1, \dots, EC'_iC''_i, \dots, C'_rC''_r)} \tag{42}$$

since the Hilbert symbols appearing in (40) are multiplicative in the first component, and hence cancel. Then according to the multiplicativity for H given in (4), the quantity (42) equals

$$\frac{F(D,E)\, F(1,1)}{F(D,1)\, F(1,E)}$$

where

$$F(D,E) = \left(\frac{DC'_i}{EC''_i}\right)^{\|\alpha_i\|^2} \left(\frac{EC''_i}{DC'_i}\right)^{\|\alpha_i\|^2} \prod_{\substack{j=1 \\ j \neq i}}^{r} \left(\frac{C'_j}{C''_j}\right)^{\|\alpha_j\|^2} \left(\frac{C''_j}{C'_j}\right)^{\|\alpha_j\|^2}$$

$$\times \prod_{j<i} \left(\frac{C'_j}{EC''_i}\right)^{2\langle\alpha_i,\alpha_j\rangle} \left(\frac{C''_j}{DC'_i}\right)^{2\langle\alpha_i,\alpha_j\rangle} \prod_{j>i} \left(\frac{DC'_i}{C''_j}\right)^{2\langle\alpha_i,\alpha_j\rangle} \left(\frac{EC''_i}{C'_j}\right)^{2\langle\alpha_i,\alpha_j\rangle}$$

$$\times \prod_{\substack{j<k \\ j,k \neq i}} \left(\frac{C'_j}{C''_k}\right)^{2\langle\alpha_j,\alpha_k\rangle} \left(\frac{C''_j}{C'_k}\right)^{2\langle\alpha_j,\alpha_k\rangle}.$$

After cancellation, this equals $\left(\frac{D}{E}\right)^{\|\alpha_i\|^2} \left(\frac{E}{D}\right)^{\|\alpha_i\|^2}$.

We have verified that the left-hand side of (40) has the same multiplicativity as Proposition 2.7 (i) predicts for the Gauss sum on the right-hand side. It remains to check that C_0 is an integer and that the term $h(D)$ agrees with $g_{\|\alpha_i\|^2}(C_0, D)$ at prime powers.

To this end, let $D = p^r$ and write $C_j = C'_j p^{k_j}$ with $p \nmid C'_j$ for $j = 1, \dots, r$. Then by definition $h(p^r)$ is

$$\frac{H(C'_1p^{k_1}, \dots, C'_ip^{k_i+r}, \dots, C'_rp^{k_r})}{H(C'_1p^{k_1}, \dots, C'_ip^{k_i}, \dots, C'_rp^{k_r})} (p^r, C'_ip^{k_i})^{\|\alpha_i\|^2} \prod_{j>i} (p^r, C'_j\, p^{k_j})^{2\langle\alpha_i,\alpha_j\rangle}.$$

Since $(p,p)=1$ we can immediately simplify the Hilbert symbols. By (4) we can simplify this as above. We need only analyze the residue symbols involving the index i as all other residue symbols immediately cancel. Hence,

$$\begin{aligned} h(p^r) &= (p^r, C_i')^{\|\alpha_i\|^2} \prod_{j>i} (p^r, C_j')^{2\langle \alpha_i,\alpha_j\rangle} \left(\frac{C_i'}{p^r}\right)^{\|\alpha_i\|^2} \left(\frac{p^r}{C_i'}\right)^{\|\alpha_i\|^2} \\ &\quad \times \prod_{j>i} \left(\frac{p^r}{C_j'}\right)^{2\langle \alpha_i,\alpha_j\rangle} \prod_{j<i} \left(\frac{C_j'}{p^r}\right)^{2\langle \alpha_i,\alpha_j\rangle} \frac{H(p^{k_1},\dots,p^{k_i+r},\dots,p^{k_r})}{H(p^{k_1},\dots,p^{k_i},\dots,p^{k_r})} \\ &= \left(\frac{(C_i')^2}{p^r}\right)^{\|\alpha_i\|^2} \prod_{j\neq i} \left(\frac{C_j'}{p^r}\right)^{2\langle \alpha_i,\alpha_j\rangle} \frac{H(p^{k_1},\dots,p^{k_i+r},\dots,p^{k_r})}{H(p^{k_1},\dots,p^{k_i},\dots,p^{k_r})} \end{aligned}$$

where we have used the reciprocity law in the final step.

Applying Proposition 5.2, we find that $H(p^{k_1},\dots,p^{k_i+r},\dots,p^{k_r})$ is non-zero for precisely two choices of r, namely $r=0$ and $r=d(w_p^{-1}\alpha_i)$. By Proposition 4.1, (41) is true, so $C_0 \in \mathfrak{o}_S$ and the two possible values of r are 0 and $\mathrm{ord}_p(C_0)+1$. On the other hand by the stability hypothesis $n/\gcd(\|\alpha_i\|^2,n) \geqslant \mathrm{ord}_p(C_0)+1$, so by Proposition 2.7, there are only two values of r such that $g_{\|\alpha_i\|^2}(C_0,p^r)\neq 0$; again, they are $r=0$ and $\mathrm{ord}_p(C_0)+1$. Thus for these two values, we have to check that

$$H(p^{k_1},\dots,p^{k_i+r},\dots,p^{k_r}) = g_{\|\alpha_i\|^2}(C_0,p^r).$$

If $r=0$, both sides are 1, so assume that $r=\mathrm{ord}_p(C_0)+1$. By (31) we have

$$\begin{aligned} h(p^r) &= \left(\frac{(C_i')^2}{p^r}\right)^{\|\alpha_i\|^2} \prod_{j\neq i} \left(\frac{C_j'}{p^r}\right)^{2\langle \alpha_i,\alpha_j\rangle} g_{\|\alpha_i\|^2}(p^{r-1},p^r) \\ &= g_{\|\alpha_i\|^2}\left((C_i')^{-2} \prod_{j\neq i} (C_j')^{-2\langle \alpha_j,\alpha_i\rangle/\langle \alpha_i,\alpha_i\rangle} p^{r-1}, p^r \right) = g_{\|\alpha_i\|^2}(C_0,p^r). \end{aligned}$$

□

Using Lemmas 5.4 and 5.5, we may rewrite the Dirichlet series $Z_\Psi(s_1,\dots,s_r)$ in terms of a Kubota Dirichlet series in the variable s_i.

Proposition 5.6. *We have*

$$Z_\Psi(s_1,\dots,s_r) =$$

$$\sum_{\substack{0\neq C_j \in \mathfrak{o}_S/\mathfrak{o}_S^\times \\ (C_1,\cdots,C_r)\ \text{admissible} \\ C_i\ i\text{-reduced}}} \mathbb{N}C_1^{-2s_1}\cdots\mathbb{N}C_r^{-2s_r} H(C_1,\dots,C_r)\, \mathcal{D}_{\|\alpha_i\|^2}(s_i, \Psi_i^{C_1,\dots,C_r}, C_0),$$

where, for fixed $C_1,\cdots,C_r$, the coefficient C_0 is defined in (39).

Proof. We have already rewritten the Dirichlet series $Z_\Psi(s_1,\dots,s_r)$ in equation (34) in terms of sums over C_j, $j=1,\dots,r$ with C_i i-reduced. The proposition then follows immediately from the previous two lemmas and the definition of $\mathcal{D}_t(s,\Psi,C)$ for S-integer C and $\Psi\in\mathcal{M}_t(\Omega)$, where $t=\|\alpha_i\|^2$. □

We are preparing to prove a functional equation corresponding to the transformation σ_i defined in (28).

Let $\mathcal{A}$ be the ring of (Dirichlet) polynomials in $q_v^{\pm 2s_1},\dots,q_v^{\pm 2s_r}$ where v runs through the finite set of places S_{fin}. Further define $\mathfrak{M}=\mathcal{A}\otimes\mathcal{M}(\Omega^r)$. We may regard elements of $\mathfrak{M}$ as functions $\Psi:\mathbb{C}^r\times(F_S^\times)^r\longrightarrow\mathbb{C}$ such that for any fixed $(s_1,\dots,s_r)\in\mathbb{C}^r$ the function

$$(C_1,C_2,\dots,C_r)\longmapsto\Psi(s_1,\dots,s_r;C_1,\dots,C_r)$$

defines an element of $\mathcal{M}(\Omega^r)$, while for any $(C_1,\dots,C_r)\in(F_S^\times)^r$, the function

$$(s_1,\dots,s_r)\longmapsto\Psi(s_1,\dots,s_r;C_1,\dots,C_r)$$

is an element of $\mathcal{A}$. We will sometimes use the notation

$$\Psi_s(C_1,\dots,C_r)=\Psi(s_1,\dots,s_r;C_1,\dots,C_r),\qquad \boldsymbol{s}=(s_1,\dots,s_r)\in\mathbb{C}^r. \tag{43}$$

We identify $\mathcal{M}(\Omega^r)$ with its image $1\otimes\mathcal{M}(\Omega^r)$ in $\mathfrak{M}$; this just consists of the $\Psi_{\boldsymbol{s}}$ that are independent of $\boldsymbol{s}\in\mathbb{C}^r$.

Recall that we defined operators σ_i on $\mathbb{C}^r$ in (28). We define corresponding operators σ_i on $\mathfrak{M}$ by

$$\begin{gathered}(\sigma_i\Psi_{\boldsymbol{s}})(C_1,\cdots,C_r)=(\sigma_i\Psi)(s_1,\cdots,s_r;C_1,\cdots,C_r)=\\ \sum_{\eta\in F_S^\times/F_S^{\times,n}}(\eta,C_i)^{\|\alpha_i\|^2}\prod_{j>i}(\eta,C_j^{2\langle\alpha_i,\alpha_j\rangle})P_{\eta C_0}(s_i)\\ \Psi(\sigma_i(s_1,\cdots,s_r);C_1,C_2,\cdots,\eta^{-1}C_i,\cdots,C_r)\end{gathered}\tag{44}$$

where, as in (39),

$$C_0=\prod_j C_j^{-2\langle\alpha_i,\alpha_j\rangle/\langle\alpha_i,\alpha_i\rangle}=C_i^{-2}\prod_{j\neq i}C_j^{-2\langle\alpha_i,\alpha_j\rangle/\langle\alpha_i,\alpha_i\rangle}.$$

Proposition 5.7. *If $\Psi\in\mathfrak{M}$, then $\sigma_i\Psi$ is in $\mathfrak{M}$.*

Proof. Let $\varepsilon_1,\dots,\varepsilon_r\in\Omega$. We have

$$\begin{gathered}(\sigma_i\Psi)(s_1,\cdots,s_r;\varepsilon_1C_1,\cdots,\varepsilon_rC_r)=\\ \sum_{\eta\in F_S^\times/F_S^{\times,n}}(\eta,\varepsilon_iC_i)^{\|\alpha_i\|^2}\prod_{j>i}(\eta,(\varepsilon_jC_j)^{2\langle\alpha_i,\alpha_j\rangle})P_{\eta\varepsilon_0C_0}(s_i)\\ \Psi(\sigma_i(s_1,\cdots,s_r);\varepsilon_1C_1,\cdots,\eta^{-1}\varepsilon_iC_i,\cdots,\varepsilon_rC_r)\end{gathered}$$

where, in parallel with the definition of C_0, we let

$$\varepsilon_0=\varepsilon_i^{-2}\prod_{j\neq i}\varepsilon_j^{-2\langle\alpha_i,\alpha_j\rangle/\langle\alpha_i,\alpha_i\rangle}.$$

Making the variable change $\eta \longmapsto \varepsilon_0^{-1}\eta$, we may rewrite the above as

$$\sum_{\eta \in F_S^\times / F_S^{\times,n}} (\varepsilon_0^{-1}\eta, \varepsilon_i C_i)^{\|\alpha_i\|^2} \prod_{j>i} (\varepsilon_0^{-1}\eta, \varepsilon_j C_j)^{2\langle \alpha_i, \alpha_j \rangle} P_{\eta C_0}(s_i)$$
$$\Psi(\sigma_i(s_1, \cdots, s_r); \varepsilon_1 C_1, \cdots, \eta^{-1}\varepsilon_0 \varepsilon_i C_i, \cdots, \varepsilon_r C_r).$$

By the transformation property (12) of the function Ψ in $\mathcal{M}(\Omega^r)$ this equals

$$\sum_{\eta \in F_S^\times / F_S^{\times,n}} (\varepsilon_0^{-1}\eta, \varepsilon_i C_i)^{\|\alpha_i\|^2} \prod_{j>i} (\varepsilon_0^{-1}\eta, \varepsilon_j C_j)^{2\langle \alpha_i, \alpha_j \rangle} P_{\eta C_0}(s_i)$$
$$(\varepsilon_0 \varepsilon_i, \eta^{-1} C_i)^{\|\alpha_i\|^2} \prod_{j \neq i} (\varepsilon_j, C_j)^{\|\alpha_j\|^2}$$
$$\prod_{j>i} (\varepsilon_0 \varepsilon_i, C_j)^{2\langle \alpha_i, \alpha_j \rangle} \prod_{j<i} (\varepsilon_j, \eta^{-1} C_i)^{2\langle \alpha_i, \alpha_j \rangle} \prod_{\substack{j < k \\ j, k \neq i}} (\varepsilon_j, C_k)^{2\langle \alpha_j, \alpha_k \rangle}$$
$$\Psi(\sigma_i(s_1, \cdots, s_r); C_1, \cdots, \eta^{-1} C_i, \cdots, C_r).$$

Looking just at factors of the form (η, ε_j) we find the following:

$$(\eta, \varepsilon_i)^{\|\alpha_i\|^2} (\varepsilon_0 \varepsilon_i, \eta^{-1})^{\|\alpha_i\|^2} \prod_{j>i} (\eta, \varepsilon_j)^{2\langle \alpha_i, \alpha_j \rangle} \prod_{j<i} (\varepsilon_j, \eta^{-1})^{2\langle \alpha_i, \alpha_j \rangle}.$$

These cancel due to the definition of ε_0. We cancel these, and remember also that due to the isotropy of Ω, we have $(\varepsilon_j, \varepsilon_k) = 1$. Rearranging the terms and using the definition of $\sigma_i \Psi$ gives

$$(\varepsilon_0^{-1}, C_i)^{\|\alpha_i\|^2} \prod_{j>i} (\varepsilon_0^{-1}, C_j)^{2\langle \alpha_i, \alpha_j \rangle} (\varepsilon_0 \varepsilon_i, C_i)^{\|\alpha_i\|^2} \prod_{j \neq i} (\varepsilon_j, C_j)^{\|\alpha_j\|^2}$$
$$\prod_{j>i} (\varepsilon_0 \varepsilon_i, C_j)^{2\langle \alpha_i, \alpha_j \rangle} \prod_{j<i} (\varepsilon_j, C_i)^{2\langle \alpha_i, \alpha_j \rangle} \prod_{\substack{j < k \\ j, k \neq i}} (\varepsilon_j, C_k)^{2\langle \alpha_j, \alpha_k \rangle}$$
$$\sigma_i \Psi_{\boldsymbol{s}}(C_1, \cdots, C_r).$$

Expanding the Hilbert symbols, everything involving ε_0 cancels and we are left with

$$(\varepsilon_i, C_i)^{\|\alpha_i\|^2} \prod_{j \neq i} (\varepsilon_j, C_j)^{\|\alpha_j\|^2} \prod_{j>i} (\varepsilon_i, C_j)^{2\langle \alpha_i, \alpha_j \rangle}$$
$$\prod_{j<i} (\varepsilon_j, C_i)^{2\langle \alpha_i, \alpha_j \rangle} \prod_{\substack{j < k \\ j, k \neq i}} (\varepsilon_j, C_k)^{2\langle \alpha_j, \alpha_k \rangle} \sigma_i \Psi_{\boldsymbol{s}}(C_1, \cdots, C_r) =$$
$$\prod_j (\varepsilon_j, C_j)^{\|\alpha_j\|^2} \prod_{j<k} (\varepsilon_j, C_k)^{2\langle \alpha_j, \alpha_k \rangle} \sigma_i \Psi_{\boldsymbol{s}}(C_1, \cdots, C_r).$$

This proves that $\sigma_i \Psi$ satisfies (12). $\square$

Let $\mathfrak{W}$ denote the group of automorphisms of $\mathfrak{M}$ generated by σ_i. This will turn out to be the group of functional equations for the multiple Dirichlet series. Each functional equation corresponding to $\sigma_i \in W$ is inherited from a functional equation for the Kubota Dirichlet series appearing in Proposition 5.6. There is a natural homomorphism $\mathfrak{W} \longrightarrow W$, which is expected to be an isomorphism, though we do not prove this. There is an action of $\mathfrak{W}$ on $\mathbb{C}^r$ induced by the action of W, and if $w \in \mathfrak{W}$ we will denote by $w\boldsymbol{s}$ the effect of w on $\boldsymbol{s} \in \mathbb{C}^r$ in this induced action.

These functional equations are formalized in the following result.

Lemma 5.8. *Given an element* $\Psi_{\boldsymbol{s}}(C_1, \ldots, C_r) \in \mathfrak{M}$, *we have*

$$\mathcal{D}^*_{\|\alpha_i\|^2}(s_i, \Psi_i^{C_1,\ldots,C_r}, C_0) = \mathbb{N}C_0^{1-2s_i} \mathcal{D}^*_{\|\alpha_i\|^2}(1 - s_i, (\sigma_i \Psi)_i^{C_1,\ldots,C_r}, C_0),$$

where $\mathcal{D}^*_{\|\alpha_i\|^2}$ *is as in (36).*

Proof. By Theorem 2.10, and using (37)

$$\begin{aligned}
&\mathcal{D}^*_{\|\alpha_i\|^2}(s_i, \Psi_i^{C_1,\ldots,C_r}, C_0) = \\
&\sum_{\eta \in F_S^\times / F_S^{\times,n}} \mathbb{N}C_0^{1-2s_i} P_{\eta C_0}^{\|\alpha_i\|^2}(s_i) \mathcal{D}^*_{\|\alpha_i\|^2}(1 - s_i, \tilde{\Psi}_{i,\eta}^{C_1,\ldots,C_r}, C_0),
\end{aligned}$$

where, in this context, by (38) and (35)

$$\begin{aligned}
\tilde{\Psi}_{i,\eta}^{C_1,\ldots,C_r}(D) &= (\eta, D)^{\|\alpha_i\|^2} \Psi_i^{C_1,\ldots,C_r}(\eta^{-1} D^{-1}) \\
&= (\eta, D)^{\|\alpha_i\|^2} (\eta D, C_i)^{\|\alpha_i\|^2} \prod_{j>i} (\eta D, C_j)^{2\langle \alpha_i, \alpha_j \rangle} \\
&\qquad \Psi(C_1, \ldots, C_{i-1}, D^{-1}\eta^{-1} C_i, C_{i+1}, \ldots, C_r).
\end{aligned}$$

If we take the s_i to have sufficiently large negative real part, we may expand the right-hand side of the above functional equation as a Dirichlet series. That is,

$$\begin{aligned}
&\sum_{\eta \in F_S^\times / F_S^{\times,n}} \mathbb{N}C_0^{1-2s_i} P_{\eta C_0}^{\|\alpha_i\|^2}(s_i)\, \mathcal{D}^*_{\|\alpha_i\|^2}(1 - s, \tilde{\Psi}_{i,\eta}^{C_1,\cdots,C_r}, C_0) = \\
&\sum_{\eta \in F_S^\times / F_S^{\times,n}} \mathbb{N}C_0^{1-2s_i} P_{\eta C_0}^{\|\alpha_i\|^2}(s_i) \sum_{0 \neq D \in \mathfrak{o}_S / \mathfrak{o}_S^\times} g_{\|\alpha_i\|^2}(C_0, D) \\
&\qquad \mathbb{N}D^{2s_i - 2} (\eta, D)^{\|\alpha_i\|^2} (D\eta, C_i)^{\|\alpha_i\|^2} \\
&\times \prod_{j>i} (D\eta, C_j)^{2\langle \alpha_i, \alpha_j \rangle} \Psi(C_1, \cdots, C_{i-1}, D^{-1}\eta^{-1} C_i, C_{i+1}, \cdots, C_r).
\end{aligned}$$

We now apply the change of variables $\eta \longrightarrow D^{-2}\eta$ (noting that $(D, D) = 1$) giving

$$\sum_{\eta \in F_S^\times / F_S^{\times,n}} \mathbb{N}C_0^{1-2s_i} P_{\eta D^{-2} C_0}^{\|\alpha_i\|^2}(s_i) \sum_{0 \neq D \in \mathfrak{o}_S / \mathfrak{o}_S^\times} g_{\|\alpha_i\|^2}(C_0, D)\, \mathbb{N}D^{2s_i - 2}$$
$$\times \left(\eta, (DC_i)^{\|\alpha_i\|^2} \prod_{j>i} C_j^{2\langle \alpha_i, \alpha_j \rangle} \right) (D, C_i)^{-\|\alpha_i\|^2} \prod_{j>i} (D, C_j)^{-2\langle \alpha_i, \alpha_j \rangle}$$
$$\times \Psi(C_1, \cdots, C_{i-1}, \eta^{-1} D C_i, C_{i+1}, \cdots, C_r).$$

Rewriting this in terms of the operator σ_i on $\mathfrak{M}$ defined in (44), we have

$$\sum_{0 \neq D \in \mathfrak{o}_S / \mathfrak{o}_S^\times} \mathbb{N}C_0^{1-2s_i} g_{\|\alpha_i\|^2}(C_0, D) \mathbb{N}D^{2s_i - 2} (D, C_i)^{-\|\alpha_i\|^2}$$
$$\prod_{j>i} (D, C_j)^{-2\langle \alpha_i, \alpha_j \rangle} (\sigma_i \Psi)(C_1, \ldots, DC_i, \ldots, C_r).$$

According to (35), this is simply

$$\sum_{0 \neq D \in \mathfrak{o}_S / \mathfrak{o}_S^\times} \mathbb{N}C_0^{1-2s_i} g_{\|\alpha_i\|^2}(C_0, D) \mathbb{N}D^{2s_i - 2} (\sigma_i \Psi)_i^{C_1, \ldots, C_r}(D) =$$
$$\mathbb{N}C_0^{1-2s_i} \mathcal{D}^*_{\|\alpha_i\|^2}(1 - s_i, (\sigma_i \Psi)_i^{C_1, \ldots, C_r}, C_0).$$

□

We can now prove the main theorem of this paper.

Theorem 5.9. *The function* $Z_\Psi^*(\mathbf{s})$ *has meromorphic continuation to the complex space* $\mathbb{C}^r$ *and satisfies the functional equation*

$$Z_{w\Psi}^*(w\boldsymbol{s}) = Z_\Psi^*(\boldsymbol{s})$$

for each $w \in \mathfrak{W}$*, where the action of* w *on* $\mathfrak{M}$ *is given by the composition of simple reflections* σ_i*. It is analytic except along the hyperplanes* $\mu(\alpha; \boldsymbol{s} - \frac{1}{2}\rho^\vee) = \frac{1}{2n(\alpha)}$*, where* α *runs through* Φ*,* $n(\alpha)$ *is defined by (22),* $\frac{1}{2}\rho^\vee = \left(\frac{1}{2}, \cdots, \frac{1}{2}\right)$ *and* μ *is defined by (25); along these hyperplanes it can have simple poles.*

Observe that the equation $\mu(-\alpha; \boldsymbol{s} - \frac{1}{2}\rho^\vee) = \frac{1}{2n(\alpha)}$ is equivalent to $\mu(\alpha; \boldsymbol{s} - \frac{1}{2}\rho^\vee) = -\frac{1}{2n(\alpha)}$, so the polar hyperplanes occur in parallel pairs.

Remark 5.10. To prove that $\mathfrak{W} \cong W$, one would need to check that σ_i and $\sigma_j \in \mathfrak{W}$ satisfy the same braid relation $(\sigma_i \sigma_j)^{r_{ij}} = 1$ as their images in W. This is not a problem because if w and $w' \in \mathfrak{W}$ correspond to the same element of w, even though we do not prove that $w\Psi = w'\Psi$, it is clear from (5.9) that $Z_{w\Psi} = Z_{w'\Psi}$ since the two Dirichlet series agree in the region of absolute convergence.

Proof. The reader may find it useful to look at the special case where $\Phi = A_2$ in [3], which is worked out in full detail.

It may be checked that the original Dirichlet series defining Z_Ψ is absolutely convergent in the region Λ_0 defined by

$$\Re(s_j) > \frac{3}{4}.$$

The expression in Proposition 5.6 is analytic in the region Λ_i which is the convex hull of $\Lambda_0 \cup \sigma_i \Lambda_0$. On this region, we claim that for each simple reflection σ_i we have

$$Z^*_{\sigma_i \Psi}(\sigma_i \boldsymbol{s}) = Z^*_\Psi(\boldsymbol{s}).$$

In order to deduce this from Proposition 5.6 and Lemma 5.8 one must consider the effect of the transformation (28). It remains to be shown that under this change of variables the quantity $\mathbb{N}C_1^{-2s_1} \cdots \mathbb{N}C_r^{-2s_r}$ gets multiplied by $\mathbb{N}C_0^{1-2s_i}$; in other words, that $\mathbb{N}C_0^{s_i - \frac{1}{2}} \cdot \mathbb{N}C_1^{-2s_1} \cdots \mathbb{N}C_r^{-2s_r}$ is invariant under (28) for the fixed index i. This follows from Proposition 4.2, since with our definitions, when $C_1, \cdots, C_r$ is admissible and C_i is i-reduced (17) implies that

$$\mathbb{N}C_1^{-2s_1} \cdots \mathbb{N}C_r^{-2s_r} = \prod_p \prod_{\alpha \in \Phi_w} M_p(-w\alpha; \boldsymbol{s})$$

and, by (41)

$$\mathbb{N}C_0 = \prod_p \mathbb{N}p^{d(w_p^{-1}\alpha_i)-1}.$$

Concerning the normalizing factor, one factor $G_{\alpha_i}(\boldsymbol{s})\zeta_{\alpha_i}(\boldsymbol{s})$ from (24) is needed to normalize $\mathcal{D}^*_{\|\alpha_i\|^2}(s_i, \Psi_i^{C_1,\ldots,C_r}, C_0)$ in Proposition 5.6; the remaining factors are permuted amongst themselves since σ_i permutes $\Phi^+ - \{\alpha_i\}$.

Arguing as in [3] we obtain analytic continuation to any simply-connected region Λ' that is a union of W-translates of the Λ_i obtained by composing functional equations. We may choose Λ' so that its convex hull is all of $\mathbb{C}^r$. The meromorphic continuation to all of $\mathbb{C}^r$ follows from Hartogs' Theorem (Hörmander [9], Theorem 2.5.10). As in [3], one actually applies Hartogs' theorem to the function

$$Z_\Psi(\boldsymbol{s}) \prod_{\alpha \in \Phi} \left(\mu\left(\alpha; \boldsymbol{s} - \frac{1}{2}\rho^\vee\right) - \frac{1}{2n(\alpha)} \right),$$

since inclusion of the factors to cancel the poles of Z_Ψ gives a function that is everywhere analytic. □

References

[1] N. Bourbaki. *Lie groups and Lie algebras. Chapters 4–6.* Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2002. Translated from the 1968 French original by Andrew Pressley.

[2] B. Brubaker and D. Bump. On Kubota's Dirichlet series. *J. Reine Angew. Math., to appear. Preprint available at* `http://sporadic.stanford.edu/bump/kubota.ps`.

[3] B. Brubaker, D. Bump, G. Chinta, S. Friedberg, and J. Hoffstein. Weyl group multiple Dirichlet series I. *Preprint, available at* `http://sporadic.stanford.edu/bump/wmd2.ps`.

[4] B. Brubaker, D. Bump, S. Friedberg, and J. Hoffstein. Weyl group multiple Dirichlet series III: Eisenstein series and twisted unstable A_2. *Preprint, available at* `http://sporadic.stanford.edu/bump/wmd3.ps`.

[5] D. Bump. *Lie groups*, volume 225 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2004.

[6] D. Bump and J. Hoffstein. Some conjectured relationships between theta functions and Eisenstein series on the metaplectic group. In *Number theory (New York, 1985/1988)*, volume 1383 of *Lecture Notes in Math.*, pages 1–11. Springer, Berlin, 1989.

[7] W. Casselman and J. Shalika. The unramified principal series of p-adic groups. II. The Whittaker function. *Compositio Math.*, 41(2):207–231, 1980.

[8] G. Chinta. Mean values of biquadratic zeta functions. *Invent. Math.*, 160(1):145–163, 2005.

[9] L. Hörmander. *An introduction to complex analysis in several variables*, volume 7 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, third edition, 1990.

[10] K. Ireland and M. Rosen. *A classical introduction to modern number theory*, volume 84 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1990.

[11] Hervé Jacquet. Fonctions de Whittaker associées aux groupes de Chevalley. *Bull. Soc. Math. France*, 95:243–309, 1967.

[12] D. A. Kazhdan and S. J. Patterson. Metaplectic forms. *Inst. Hautes Études Sci. Publ. Math.*, (59):35–142, 1984.

[13] T. Kubota. *On automorphic functions and the reciprocity law in a number field.* Lectures in Mathematics, Department of Mathematics, Kyoto University, No. 2. Kinokuniya Book-Store Co. Ltd., Tokyo, 1969.

[14] J. Neukirch. *Algebraic number theory*, volume 322 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999. Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder.

[15] S. J. Patterson. Whittaker models of generalized theta series. In *Seminar on number theory, Paris 1982–83 (Paris, 1982/1983)*, volume 51 of *Progr. Math.*, pages 199–232. Birkhäuser Boston, Boston, MA, 1984.

[16] A. Selberg. A new type of zeta functions connected with quadratic forms. In *Report of the Institute in the Theory of Numbers*, pages 207–210. University of Colorado, Boulder, Colorado, 1959.

[17] T. A. Springer. Reductive groups. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 3–27. Amer. Math. Soc., Providence, R.I., 1979.

[18] T. Suzuki. Some results on the coefficients of the biquadratic theta series. *J. Reine Angew. Math.*, 340:70–117, 1983.

[19] T. Suzuki. Rankin-Selberg convolutions of generalized theta series. *J. Reine Angew. Math.*, 414:149–205, 1991.

[20] T. Suzuki. On the biquadratic theta series. *J. Reine Angew. Math.*, 438:31–85, 1993.