GEOMETRIC LANGLANDS DUALITY AND REPRESENTATIONS OF ALGEBRAIC GROUPS OVER COMMUTATIVE RINGS

I. MIRKOVIĆ AND K. VILONEN

Contents

1. **Introduction**

In this paper we give a geometric version of the Satake isomorphism. Given a connected complex reductive algebraic group G, we show that the category of representations of $\check{G}$, the Langlands dual of G, is naturally equivalent to a certain category of perverse sheaves on the complex affine Grassmannian of G. The crucial point is that we prove this equivalence over commutative rings, instead of just $\mathbb{C}$.

We now give a more precise version of our result. Let G be a reductive algebraic group over the complex numbers. We write $G_{\mathcal{O}}$ for the group scheme $G(\mathbb{C}[[z]])$ and $\mathcal{G}r$ for the affine Grassmannian of $G(\mathbb{C}((z)))/G(\mathbb{C}[[z]])$; the affine Gramannian is an ind-scheme, i.e., a direct limit of schemes. Let $\Bbbk$ be a Noetherian, commutative unital ring of finite global dimension. One can imagine $\Bbbk$ to be $\mathbb{C}$, $\mathbb{Z}$, or $\bar{\mathbb{F}}_q$, for example. Let us write $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}r, \Bbbk)$ for the category of $G_{\mathcal{O}}$-equivariant perverse sheaves with $\Bbbk$-coefficients. Furthermore, let $\mathrm{Rep}_{\check{G}_{\Bbbk}}$ stand for the category $\Bbbk$-representations of $\check{G}_{\Bbbk}$;

Date: January 14, 2004.

here $\check{G}_{\Bbbk}$ denotes the canonical smooth split reductive group scheme over $\Bbbk$ whose root datum is dual to that of G. The goal of this paper is to prove the following:

$$\text{the categories } \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r},\Bbbk) \text{ and } \mathrm{Rep}_{\check{G}_{\Bbbk}} \text{ are equivalent as tensor categories}. \tag{1.1}$$

We do slightly more than this. We give a canonical construction of the of the group scheme $\check{G}_{\Bbbk}$ in terms of $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r},\Bbbk)$. In particular, we give a canonical construction of the Chevalley group scheme $\check{G}_{\mathbb{Z}}$. This is one way to view our theorem. We can also view it as giving a geometric interpretation of representation theory of algebraic groups over commutative rings. Although our results yield an interpretation of representation theory over arbitrary commutative rings, note that on the geometric side we work over the complex numbers and use the classical topology. The advantage of the classical topology is that one can work with sheaves with coefficients in arbitrary commutative rings, in particular, we can use integer coefficients. Finally, our work can be viewed as providing the unramified local geometric Langlands correspondence. In this context it crucial that one works on the geometric side also over fields other than $\mathbb{C}$. The modifications needed to do so are explained in section 14. This can then be used to give a definition of the notion of an automorphic sheaf to be a Hecke eigensheaf. Note that arbitrary systems of coefficients are allowed here.

We describe the contents of the paper briefly. Section 2 is devoted to the basic definitions involving the affine Grassmannian and the notion of perverse sheaves that we adopt. In section 3 we introduce our main tool, the weight functors. In this section we also give our crucial dimension estimates, use them to prove the exactness of the weight functors, and, finally, we decompose the global cohomology functor into a direct sum of the weight functors. The next section 4 is devoted to putting a tensor structure on the category $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r},\Bbbk)$; here, again, we make use of the dimension estimates of the previous section. In section 5 we give, using the Beilinson-Drinfeld Grassmannian, a commutativity constraint on the tensor structure. In section 6 we show that global cohomology is a tensor functor and we also show that it is tensor functor in the weighted sense. Section 7 is devoted to the simpler case when $\Bbbk$ is a field of characteristic zero. The next section 8 treats standard sheaves and we show that their cohomology is given by specific algebraic cycles providing their cohomology with a canonical basis. In the next section 9 we prove that the weight functors introduced in section 3 are representable. This, then, will provide us with a supply of projective objects. In section 10 we study the structure of these projectives and prove that they have filtrations whose associated graded consists of standard sheaves. In section 11 we show that $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r},\Bbbk)$ is equivalent, as a tensor category, to $\mathrm{Rep}_{\tilde{G}_{\Bbbk}}$ for some group scheme $\tilde{G}_{\Bbbk}$. Then, in the next section 12, we identify $\tilde{G}_{\Bbbk}$ with $\check{G}_{\Bbbk}$. A crucial ingredient in this section is the work of Prasad and Yu [PY]. We then briefly discuss in section 13 our results from the point of view of representation theory. In the final section 14 we briefly indicate how our arguments have to be modified to work in the etale topology.

Most of the results in this paper appeared in the announcement [MiV2]. These questions were previously treated in [Gi] in the case $\Bbbk = \mathbb{C}$. Since our announcement, the paper [Na] has appeared. Certain technical points that are necessary for us are treated in [Na]. Instead of repeating the discussion here, we have chosen to refer to [Na] instead.

We thank the MPI in Bonn, where some of this research was carried out. Furthermore, we thank Prasad and Yu for answering our questions in the form of the paper [PY].

2. Perverse sheaves on the affine Grassmannian

We begin this section by recalling the construction and the basic properties of the affine Grassmannian $\mathcal{G}\mathrm{r}$. For proofs of these facts we refer to §4.5 of [BD]. See also, [BL1] and [BL2]. Then we introduce the main object of study, the category $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}, \Bbbk)$ of equivariant perverse sheaves on $\mathcal{G}\mathrm{r}$.

Let G be a complex, connected, reductive algebraic group. We write $\mathcal{O}$ for the formal power series ring $\mathbb{C}[[z]]$ and $\mathcal{K}$ for its fraction field $\mathbb{C}((z))$. Let $G(\mathcal{K})$ and $G(\mathcal{O})$ denote, as usual, the sets of the $\mathcal{K}$-valued and the $\mathcal{O}$-valued points of G, respectively. The affine Grassmannian is defined as the quotient $G(\mathcal{K})/G(\mathcal{O})$. The sets $G(\mathcal{K})$ and $G(\mathcal{O})$, and the quotient $G(\mathcal{K})/G(\mathcal{O})$ have an algebraic structure over the complex numbers. The space $G(\mathcal{O})$ has a structure of a group scheme, denoted by $G_{\mathcal{O}}$, over $\mathbb{C}$ and the spaces $G(\mathcal{K})$ and $G(\mathcal{K})/G(\mathcal{O})$ have structures of ind-schemes which we denote by $G_{\mathcal{K}}$ and $\mathcal{G}\mathrm{r}$, respectively. For us an ind-scheme means a direct limit of family of schemes where all the maps are closed embeddings. The morphism $\pi : G_{\mathcal{K}} \to \mathcal{G}\mathrm{r}$ is locally trivial in the Zariski topology, i.e., there exists a Zariski open subset $U \subset \mathcal{G}\mathrm{r}$ such that $\pi^{-1}(U) \cong U \times G_{\mathcal{O}}$ and π restricted to $U \times G_{\mathcal{O}}$ is simply projection to the first factor. For details see for example [BL1, LS]. We write $\mathcal{G}\mathrm{r}$ as a limit

$$\mathcal{G}\mathrm{r} \;=\; \varinjlim \mathcal{G}\mathrm{r}_n\,, \tag{2.1}$$

where the $\mathcal{G}\mathrm{r}_n$ are finite dimensional schemes which are $G_{\mathcal{O}}$-invariant. The group $G_{\mathcal{O}}$ acts on the $\mathcal{G}\mathrm{r}_n$ via a finite dimensional quotient.

In this paper, with the exception of section 14, we consider sheaves in the classical topology. As we work with topological sheaves, it suffices for us to consider the spaces $G_{\mathcal{O}}$, $G_{\mathcal{K}}$, and $\mathcal{G}\mathrm{r}$ as varieties, i.e., as reduced schemes. We will do so for the rest of the paper.

If $G = T$ is torus of rank r then, as a reduced ind-scheme, $\mathcal{G}\mathrm{r} \cong X_*(T) = \mathrm{Hom}(\mathbb{C}^*, T)$, i.e., in this case the loop Grassmannian is discrete. Note that, because T is abelian, the loop Grassmannian is an group ind-scheme. Let G be a reductive group, write $Z(G)$ for the center of G and let $\bar{T} = Z(G)^0$ denote connected component of the center. Let us further set $\bar{G} = G\bar{T}$. Then, as is easy to see, the map $\mathcal{G}\mathrm{r} \to \mathcal{G}\mathrm{r}_{\bar{G}}$ is a trivial covering with covering group $X_*(\bar{T}) = \mathrm{Hom}(\mathbb{C}^*, \bar{T})$, i.e., $\mathcal{G}\mathrm{r} \cong \mathcal{G}\mathrm{r}_{\bar{G}} \times X_*(\bar{T})$, non-canonically. Note also that the connected components of $\mathcal{G}\mathrm{r}$ are exactly parameterized by the component group of $G_{\mathcal{K}}$, i.e., by $G_{\mathcal{K}}/(G_{\mathcal{K}})^0$. This latter group is isomorphic to $\pi_1(G)$, the topological fundamental group of G.

The group scheme $G_{\mathcal{O}}$ acts on $\mathcal{G}\mathrm{r}$ with finite dimensional orbits. In order to describe the orbit structure, let us fix a maximal torus $T \subset G$. We write W for the Weyl group and $X_*(T)$ for the coweights $\mathrm{Hom}(\mathbb{C}^*, T)$. Then the $G_{\mathcal{O}}$-orbits on $\mathcal{G}\mathrm{r}$ are parameterized by the W-orbits in $X_*(T)$, and given $\lambda \in X_*(T)$ the $G_{\mathcal{O}}$-orbit associated to it is $\mathcal{G}\mathrm{r}^\lambda = G_{\mathcal{O}} \cdot L_\lambda \subset \mathcal{G}\mathrm{r}$, where L_λ denotes the image of the point $\lambda \in X_*(T)$ in $\mathcal{G}\mathrm{r}$. Note that the points L_λ are precisely the T-fixed points in the Grassmannian. To describe the closure relation between the $G_{\mathcal{O}}$-orbits, we choose a Borel $B \supset T$, i.e., we make a choice of positive roots. Then, for dominant λ and μ we have

$$\mathcal{G}\mathrm{r}^{\mu} \subset \overline{\mathcal{G}\mathrm{r}^{\lambda}} \quad \text{if and only if} \quad \lambda - \mu \quad \text{is a sum of positive coroots}. \tag{2.2}$$

In a few arguments in this paper it will be important for us to consider a Kac-Moody group associated to the loop group $G_{\mathcal{K}}$. Let us write $\Phi = \Phi(G,T)$ for the the root system of G with respect to T, and we write similarly $\check{\Phi} = \check{\Phi}(G,T)$ for the coroots. Let $\Gamma \cong \mathbb{C}^*$ denote the subgroup of automorphisms of $\mathcal{K}$ which acts by multiplying the parameter t, as follows: $s \cdot f(z) = f(s^{-1}z), s \in \mathbb{C}^* \cong \Gamma$. The group Γ acts on $G_{\mathcal{O}}$ and $G_{\mathcal{K}}$ and hence we can form the the semi-direct product $\tilde{G}_{\mathcal{K}} = G_{\mathcal{K}} \rtimes \Gamma$. A Kac-Moody group is a central extension, by the multiplicative group, of $\tilde{G}_{\mathcal{K}}$. We fix a Borel subgroup B with unipotent radical N and a Cartan subgroup $T \subseteq B = TN \subseteq G$. Then $\tilde{T} = T \times \Gamma$ is a Cartan subgroup of $\tilde{G}_{\mathcal{K}}$. Let us write $\delta \in X^*(\tilde{T})$ for cocharacter which is trivial on T and identity on the factor $\Gamma \cong \mathbb{C}^*$ and let $\check{\delta} \in X_*(\tilde{T})$ stand for the composition $\mathbb{C}^* \cong \Gamma \subset T \times \Gamma = \tilde{T}$. We also view the roots Φ as characters on $\tilde{T}$, which are trivial on Γ. The $\tilde{T}$-eigenspaces in $\mathfrak{g}_{\mathcal{K}}$ are given by

$$(\mathfrak{g}_{\mathcal{K}})_{k\delta+\alpha} \overset{def}{=} z^k \mathfrak{g}_{\alpha}, \;\; k \in \mathbb{Z}, \alpha \in \Phi \cup \{0\}, \tag{2.3}$$

and thus the roots of $G_{\mathcal{K}}$ are given by $\tilde{\Phi} = \{\alpha + k\delta \in X^*(\tilde{T}) \mid \alpha \in \Phi \cup \{0\}, k \in \mathbb{Z}\} - \{0\}$.

Furthermore, the orbit $G \cdot L_\lambda$ is isomorphic to the flag manifold G/P_λ, where P_λ is the parabolic associated to the roots $\{\alpha \in \Phi \mid \lambda(\alpha) = 0\}$. The orbit $\mathcal{G}\mathrm{r}^{\lambda}$ can be viewed as a G-equivariant vector bundle over G/P_λ. One way to see this is to observe that the varieties $G \cdot L_\lambda$ are the fixed point sets of the $\mathbb{G}_m$-action via the cocharacter $\check{\delta}$. In this language,

$$\mathcal{G}\mathrm{r}^{\lambda} \;=\; \{x \in \mathcal{G}\mathrm{r} \mid \lim_{s \to 0} \check{\delta}(s)x \in G \cdot L_\lambda\} \tag{2.4}$$

In particular, the orbits $\mathcal{G}\mathrm{r}^{\lambda}$ are simply connected. If we choose a Borel B containing T and if we choose the parameter $\lambda \in X_*(T)$ of the orbit $\mathcal{G}\mathrm{r}^{\lambda}$ to be dominant, then $\dim(\mathcal{G}\mathrm{r}^{\lambda}) = 2\rho(\lambda)$, where we have written $\rho(\lambda)$ for the height of λ with respect to ρ, i.e., $\rho(\lambda) = \lambda(\rho)$, where ρ, as usual, is half the sum of positive roots (positive roots are the ones in B). Let us consider the map $\mathrm{ev}_0 : G_{\mathcal{O}} \to G$, evaluation at zero. We write $I = {\mathrm{ev}_0}^{-1}(B)$ for the Iwahori subgroup and $K = {\mathrm{ev}_0}^{-1}(0)$ for the highest congruence subgroup. The I-orbits are parameterized by $X_*(T)$, and because the I-orbits are also ${\mathrm{ev}_0}^{-1}(N)$-orbits, they are are affine spaces. This way each $G_{\mathcal{O}}$-orbit acquires a cell decomposition as a union of I-orbits. The K-orbit $K \cdot L_\lambda$ is the fiber of the vector bundle $\mathcal{G}\mathrm{r}^{\lambda} \to G/P_\lambda$. Let us consider the sub ind group scheme $G_{\mathcal{O}}^-$ of $G_{\mathcal{K}}$ whose $\mathbb{C}$-points consist of $G(\mathbb{C}[z^{-1}])$. The $G_{\mathcal{O}}^-$-orbits are also indexed by W-orbits in $X_*(T)$ and the orbit attached to $\lambda \in X_*(T)$ is $G_{\mathcal{O}}^- \cdot L_\lambda$. The $G_{\mathcal{O}}^-$-orbits are opposite to the $G_{\mathcal{O}}$-orbits in the following sense:

$$G_{\mathcal{O}}^- \cdot L_\lambda \;=\; \{x \in \mathcal{G}\mathrm{r} \mid \lim_{s \to \infty} \check{\delta}(s)x \in G \cdot L_\lambda\}. \tag{2.5}$$

The above description implies that

$$(G_{\mathcal{O}}^- \cdot L_\lambda) \cap \overline{\mathcal{G}\mathrm{r}^{\lambda}} \;=\; G \cdot L_\lambda \tag{2.6}$$

The group $G_{\mathcal{O}}^-$ contains a negative level congruence subgroup K_- which is the kernel of the evaluation map $G(\mathbb{C}[z^{-1}]) \to G$ at infinity. Just as for $G_{\mathcal{O}}$, the fiber of the projection $G_{\mathcal{O}}^- \cdot L_\lambda \to G/P_\lambda$ is $K_- \cdot L_\lambda$.

We will recall briefly the notion of perverse sheaves that we will use in this paper [BBD]. Let X be a complex algebraic variety with a fixed (Whitney) stratification $\mathcal{S}$. We also fix a commutative, unital ring $\Bbbk$. For simplicity of exposition we assume that $\Bbbk$ is Noetherian of finite global dimension. We denote by $\mathrm{D}_{\mathcal{S}}(X,\Bbbk)$ the bounded $\mathcal{S}$-constructible derived category of $\Bbbk$-sheaves. This is the full subcategory of the derived category of $\Bbbk$-sheaves on X whose objects $\mathcal{F}$ satisfy the following two conditions:

$$
\begin{aligned}
&i) \ \ \mathrm{H}^k(X,\mathcal{F}) = 0 \quad \text{for } |k| > 0\,,\\
&ii) \ \ \mathrm{H}^k(\mathcal{F})\big|S \quad \text{is a local system of finitely generated } \Bbbk\text{-modules}\\
&\qquad \text{for all } S \in \mathcal{S}\,.
\end{aligned}
$$

As usual we define the full subcategory $\mathrm{P}_{\mathcal{S}}(X,\Bbbk)$ of perverse sheaves as follows. An $\mathcal{F} \in \mathrm{D}_{\mathcal{S}}(X,\Bbbk)$ is perverse if the following two conditions are satisfied:

$$
\begin{aligned}
&i) \ \ \mathrm{H}^k(i^*\mathcal{F}) = 0 \quad \text{for } k > -\dim_{\mathbb{C}} S \ \text{ for any } \ i : S \hookrightarrow X\,, S \in \mathcal{S}\,,\\
&ii) \ \ \mathrm{H}^k(i^!\mathcal{F}) = 0 \quad \text{for } k < \dim_{\mathbb{C}} S \ \text{ for any } \ i : S \hookrightarrow X\,, S \in \mathcal{S}\,.
\end{aligned}
$$

As is explained in [BBD], perverse sheaves $\mathrm{P}_{\mathcal{S}}(X,\Bbbk)$ form an abelian category and there is a cohomological functor

$$
{}^p\mathrm{H}^0 : \mathrm{D}_{\mathcal{S}}(X,\Bbbk) \to \mathrm{P}_{\mathcal{S}}(X,\Bbbk)\,.
$$

Given a stratum $S \in \mathcal{S}$ and M a finitely generated $\Bbbk$-module then Rj_*M and $j_!M$ belong in $\mathrm{D}_{\mathcal{S}}(X,\Bbbk)$. Following [BBD] we write ${}^pj_*M$ for ${}^p\mathrm{H}^0(Rj_*M) \in \mathrm{P}_{\mathcal{S}}(X,\Bbbk)$ and ${}^pj_!M$ for ${}^p\mathrm{H}^0(j_!M) \in \mathrm{P}_{\mathcal{S}}(X,\Bbbk)$. We use this type of notation systematically throughout the paper. If $Y \subset X$ is locally closed and is a union of strata in $\mathcal{S}$ then, by abuse of notation, we denote by $\mathrm{P}_{\mathcal{S}}(Y,\Bbbk)$ the category $\mathrm{P}_{\mathcal{T}}(Y,\Bbbk)$, where $\mathcal{T} = \{S \in \mathcal{S} \mid S \subset Y\}$.

Let us now assume that we have an action of a connected algebraic group K on X, given by $a : K \times X \to X$. Fix a Whitney stratification $\mathcal{S}$ of X such that the action of K preserves the strata. Recall that an $\mathcal{F} \in \mathrm{P}_{\mathcal{S}}(X,\Bbbk)$ is said to be K-equivariant if there exists an isomorphism $\phi : a^*\mathcal{F} \cong p^*\mathcal{F}$ such that $\phi\big|\{1\} \times X = \mathrm{id}$. Here $p : K \times X \to X$ is the projection to the second factor. If such an isomorphism ϕ exists it is unique. We denote by $\mathrm{P}_K(X,\Bbbk)$ the full subcategory of $\mathrm{P}_{\mathcal{S}}(X,\Bbbk)$ consisting of equivariant perverse sheaves. In a few instances we also make use of the equivariant derived category $\mathrm{D}_K(X,\Bbbk)$, see [BL].

Let us now return to our situation. Denote the stratification induced by the $G_{\mathcal{O}}$-orbits on the Grassmannian $\mathcal{G}\mathrm{r}$ by $\mathcal{S}$. The closed embeddings $\mathcal{G}\mathrm{r}_n \subset \mathcal{G}\mathrm{r}_m$, for $n \leq m$ induce embeddings of categories $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}_n,\Bbbk) \to \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}_m,\Bbbk)$. This allows us to define the category of $G_{\mathcal{O}}$-equivariant perverse sheaves on $\mathcal{G}\mathrm{r}$ as

$$
\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r},\Bbbk) \ =_{\mathrm{def}} \ \varinjlim \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}_n,\Bbbk)\,.
$$

Similarly we define $\mathrm{P}_{\mathcal{S}}(\mathcal{G}\mathrm{r},\Bbbk)$, the category of perverse sheaves on $\mathcal{G}\mathrm{r}$ which are constructible with respect to the $G_{\mathcal{O}}$-orbits. In our setting we have

2.1. **Proposition.** *The categories* $\mathrm{P}_{\mathcal{S}}(\mathcal{G}r,\Bbbk)$ *and* $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}r,\Bbbk)$ *are naturally equivalent.*

We give a proof of this proposition in appendix A; the proof makes use of results of section 3.

Let us write $\mathrm{Aut}(\mathcal{O})$ for the group of automorphisms of the formal disc $\mathrm{Spec}(\mathcal{O})$. The group scheme $\mathrm{Aut}(\mathcal{O})$ acts on $G_{\mathcal{K}}$, $G_{\mathcal{O}}$, and $\mathcal{G}\mathrm{r}$. This action and the action of $G_{\mathcal{O}}$ on the affine Grassmannian extend to an action of the semidirect product $G_{\mathcal{O}} \rtimes \mathrm{Aut}(\mathcal{O})$ on $\mathcal{G}\mathrm{r}$. In the appendix A we also prove

2.2. **Proposition.** *The categories* $\mathrm{P}_{G_{\mathcal{O}} \rtimes \mathrm{Aut}(\mathcal{O})}(\mathcal{G}r, \Bbbk)$ *and* $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}r, \Bbbk)$ *are naturally equivalent.*

2.3. **Remark.** *If* $\Bbbk$ *is field of characteristic zero then propositions 2.1 and 2.2 follow immediately from lemma 7.1.*

Finally, we fix some notation that will be used throughout the paper. Given a $G_{\mathcal{O}}$-orbit $\mathcal{G}\mathrm{r}_\lambda$, $\lambda \in X_*(T)$, and a $\Bbbk$-module M we write $\mathcal{I}_!(\lambda, M)$, $\mathcal{I}_*(\lambda, M)$, and $\mathcal{I}_{!*}(\lambda, M)$ for the perverse sheaves ${}^p j_!(M[\dim(\mathcal{G}\mathrm{r}^\lambda)])$, $j_{!*}(M[\dim(\mathcal{G}\mathrm{r}^\lambda)])$, and ${}^p j_*(M[\dim(\mathcal{G}\mathrm{r}^\lambda)])$, respectively; here $j : \mathcal{G}\mathrm{r}^\lambda \to \mathcal{G}\mathrm{r}$ denotes the inclusion.

3. Semi-infinite orbits and weight functors

Recall that we have fixed a maximal torus T, a Borel $B \supset T$ and denoted by N the unipotent radical of B. Furthermore, we write $N_{\mathcal{K}}$ for the group ind-subscheme of $G_{\mathcal{K}}$ whose $\mathbb{C}$-points are $N(\mathcal{K})$. The $N_{\mathcal{K}}$-orbits on $\mathcal{G}\mathrm{r}$ are parameterized by $X_*(T)$; to each $\nu \in X_*(T) = \mathrm{Hom}(\mathbb{C}^*, T)$ we associate the $N_{\mathcal{K}}$-orbit $S_\nu =_{\mathrm{def}} N_{\mathcal{K}} \cdot L_\nu$. Note that these orbits are neither of finite dimension nor of finite codimension. We view them as ind-varieties, in particular, their intersection with any $\overline{\mathcal{G}\mathrm{r}^\lambda}$ is an algebraic variety. The following proposition gives the basic properties of these orbits. Recall that for $\mu, \lambda \in X_*(T)$ we say that $\mu \le \lambda$ if $\lambda - \mu$ is a sum of positive coroots.

3.1. **Proposition.** *We have*

(a) $\overline{S_\nu} = \cup_{\eta \le \nu} S_\eta$.

(b) Inside $\overline{S_\nu}$*, the boundary of* S_ν *is given by a hyperplane section under an embedding of* $\mathcal{G}r$ *in projective space.*

Proof. Because translation by elements in $T_{\mathcal{K}}$ is an automorphism of the Grassmannian, it suffices to prove the claim on the identity component of the Grassmannian. Hence, we may assume that G is simply connected. In that case G is a product of simple factors and we may then furthermore assume that G is simple and simply connected. Let us index the vertices of the the Dynkin diagram by I. Then I can also be used to index the simple roots and coroots. For a simple coroot $\check{\alpha}_i$, $i \in I$, there is T-stable $\mathbb{P}^1$ passing through $L_{\nu - \check{\alpha}_i}$ such that the remaining $\mathbb{A}^1$ lies in S_ν, constructed as follows. Let $k = \alpha_i(\nu)$, then $\psi = \alpha_i + k\delta \in X^*(\Gamma \times T)$ is a root in $\mathfrak{g}_{\mathcal{K}}$ with the root space $z^k \mathfrak{g}_{\alpha_i}$. Let us consider the SL_2-subgroup $S \subset G_{\mathcal{K}}$ generated by one parameter subgroups U_ψ and $U_{-\psi}$ associated to the roots ψ and $-\psi$. Because the group $U_{-\psi}$ fixes L_ν and U_ψ does not, we conclude that the orbit $S \cdot L_\nu$ is a $\mathbb{P}^1$, and that $\mathbb{A}^1 \cong U_\psi \cdot L_\nu \subset S_\nu$. The point at infinity is thus gotten by applying the element in $S \cong SL_2$ which switches the root spaces to L_ν. An easy calculation then yields $L_{\nu - \check{\alpha}_i}$ as the point at infinity. Hence $S_{\nu - \check{\alpha}_i} \in \overline{S_\nu}$ for any simple coroot $\check{\alpha}_i$ and therefore $\cup_{\eta \le \nu} S_\eta \subset \overline{S_\nu}$.

To prove the rest of the proposition we embed the ind-variety $\mathcal{G}r$ in an ind-projective space $\mathbb{P}(V)$ via an ample line bundle $\mathcal{L}$ on $\mathcal{G}r$. For simplicity we choose $\mathcal{L}$ to be the positive generator of the Picard group of $\mathcal{G}r$. The action of $G_{\mathcal{K}}$ on $\mathcal{G}r$ only extends to a projective action on the line bundle $\mathcal{L}$. To get an action on $\mathcal{L}$ we must pass to the Kac-Moody group $\hat{G}_{\mathcal{K}}$ associated to $G_{\mathcal{K}}$, [LS]. The highest weight Λ_0 of the resulting representation $V = \mathrm{H}^0(\mathcal{G}r, \mathcal{L})$ is zero on T and one on the central $\mathbb{G}_m$. Thus we get a $G_{\mathcal{K}}$-equivariant embedding $\phi : \mathcal{G}r \hookrightarrow \mathbb{P}(V)$ which maps L_0 to the highest weight line V_{Λ_0}. In particular, the T-weight of the line $\phi(L_0) = V_{\Lambda_0}$ is zero.

We need a formula for the T-weight of the line $\phi(L_\nu) = \nu \cdot \phi(L_0) = \nu \cdot V_{\Lambda_0}$. Now, $\nu \cdot V_{\Lambda_0} = V_{\tilde{\nu}\cdot\Lambda_0}$, where $\tilde{\nu}$ is any lift of the element $\nu \in X_*(T)$ to $\hat{T}_{\mathcal{K}}$, the restriction of the central extension $\hat{G}_{\mathcal{K}}$ of $\tilde{G}_{\mathcal{K}}$ by $\mathbb{G}_m$ to $\tilde{T}_{\mathcal{K}}$. For $t \in T$,

$$(\tilde{\nu} \cdot \Lambda_0)(t) = \Lambda_0(\tilde{\nu}^{-1} t \tilde{\nu}) = \Lambda_0(\tilde{\nu}^{-1} t \tilde{\nu} t^{-1}) \,, \tag{3.1}$$

since $\Lambda_0(t) = 1$. The commutator $x, y \mapsto xyx^{-1}y^{-1}$ on $\hat{T}_{\mathcal{K}}$ descends to a bilinear pairing of $T_{\mathcal{K}} \times T_{\mathcal{K}}$ to the central $\mathbb{G}_m$. The restriction of this pairing, $X_*(T) \times T \to \mathbb{G}_m$, can be viewed as a homomorphism $\iota : X_*(T) \to X^*(T)$, or, equivalently, as a bilinear form $(\ ,\ )_*$ on $X_*(T)$. Since Λ_0 is identity on the central $\mathbb{G}_m$ and since $\tilde{\nu}^{-1} t \tilde{\nu} t^{-1} \in \mathbb{G}_m$, we see that

$$\tilde{\nu} \cdot \Lambda_0(t) \;=\; \tilde{\nu}^{-1} t \tilde{\nu} t^{-1} \;=\; (\iota\nu)(t)^{-1} \,, \tag{3.2}$$

i.e., $\tilde{\nu} \cdot \Lambda_0 = -\iota\nu$ on T. We will now describe the morphism ι.

The description of the central extension of $\tilde{\mathfrak{g}}_{\mathcal{K}}$, corresponding to $\hat{G}_{\mathcal{K}}$, makes use of an invariant bilinear form $(\ ,\ )$ on $\mathfrak{g}$, see, for example, [PS]. From the basic formula for the coadjoint action of $\hat{G}_{\mathcal{K}}$ (see, for example, [PS]), it is clear that the form $(\ ,\ )_*$ above is the restriction of $(\ ,\ )$ to $\mathfrak{t} = \mathbb{C} \otimes X_*(T)$. The form $(\ ,\ )$ is characterized by the property that the corresponding bilinear form $(\ ,\ )^*$ on $\mathfrak{t}^*$ satisfies $(\theta,\theta)^* = 2$ for the longest root θ. Now, for a root $\alpha \in \Phi$ we find that

$$\iota\check{\alpha} \;=\; \frac{2}{(\alpha,\alpha)^*}\alpha \;=\; \frac{(\theta,\theta)^*}{(\alpha,\alpha)^*}\alpha \in \{1,2,3\} \cdot \alpha \tag{3.3}$$

We conclude that $\iota(\mathbb{Z}\check{\Phi}) \cap \mathbb{Z}_+\Phi_+ = \iota(\mathbb{Z}_+\check{\Phi}_+)$, i.e.,

$$\nu < \eta \quad \text{is equivalent to} \quad \iota\nu < \iota\eta \qquad \text{for } \nu, \eta \in X_*(T) \,. \tag{3.4}$$

Let us write $V_{>-\iota\nu}$ for the sum of all the T-weight spaces of V whose T-weight is bigger than $-\iota\nu$ and $V_{\geq -\iota\nu}$ for the sum of all the T-weight spaces of V whose T-weight is bigger than or equal to $-\iota\nu$. Clearly the central extension of $N_{\mathcal{K}}$ acts by increasing the T weights, i.e., its action preserves the subspaces $V_{>-\iota\nu}$ and $V_{\geq -\iota\nu}$. This, together with (3.4), implies that $\cup_{\eta \leq \nu} S_\eta = \phi^{-1}(V_{\geq -\iota\nu})$. In particular, $\cup_{\eta\leq\nu} S_\eta$ is closed. This, with $\cup_{\eta\leq\nu} S_\eta \subset \overline{S_\nu}$, implies that $\overline{S_\nu} = \cup_{\eta\leq\nu} S_\eta$, proving part (a) of the proposition.

To prove part (b), we first observe that $\cup_{\eta<\nu} S_\eta = \phi^{-1}(V_{>-\iota\nu})$. The line $\phi(L_\nu) \subset V_{\geq-\iota\nu} - V_{>-\iota\nu}$. Let us choose a linear form f on V which is non-zero on the line $\phi(L_\nu)$ and which vanishes on all T-eigenspaces whose eigenvalue is different from $-\iota\nu$. Let us write H for the hyperplane $\{f = 0\} \subseteq V$. By construction, for $v \in \phi(L_\nu)$, and n in the central extension of $N_{\mathcal{K}}$, $nv \in \mathbb{C}^* \cdot v + V_{>-\iota\nu}$. So $v \neq 0$ implies $f(nv) \neq 0$, and we see that $S_\nu \cap H = \emptyset$. Since $\cup_{\eta<\nu} S_\eta \subset H$, we conclude that $\overline{S_\nu} \cap H = \cup_{\eta<\nu} S_\eta$, as required.

□

Let us also consider the unipotent radical $\bar{N}$ of the Borel $\bar{B}$ opposite to B. The $\bar{N}_{\mathcal{K}}$-orbits on $\mathcal{G}\mathrm{r}$ are again parameterized by $X_*(T)$: to each $\nu \in X_*(T)$ we associate the orbit $T_\nu = \bar{N}_{\mathcal{K}} \cdot L_\nu$. The orbits S_ν and T_ν intersect the orbits $\mathcal{G}\mathrm{r}^\lambda$ as follows:

3.2. Theorem. *We have*

a) The intersection $S_\nu \cap \mathcal{G}r^\lambda$ is non-empty precisely when $L_\nu \in \overline{\mathcal{G}r^\lambda}$ and then it is of pure dimension $\rho(\nu+\lambda)$, if λ is chosen dominant.

b) The intersection $T_\nu \cap \mathcal{G}r^\lambda$ is non-empty precisely when $L_\nu \in \overline{\mathcal{G}r^\lambda}$ and then it is of pure dimension $-\rho(\nu+\lambda)$, if λ is chosen anti-dominant.

3.3. Remark. *Note that, by (2.2), $L_\nu \in \overline{\mathcal{G}r^\lambda}$ if and only if ν is a weight of the the irreducible representation of $\check{G}$ of highest weight λ; here $\check{G}$ is the complex Langlands dual group of G, i.e., the complex reductive group whose root datum is dual to that of G.*

Proof. It suffices to prove the statement a). Let the coweight $2\check{\rho} : \mathbb{G}_m \to T$ be the sum of positive coroots. When we act by conjugation by this coweight on $N_{\mathcal{K}}$, we see that for any element $n \in N_{\mathcal{K}}$, $\lim_{s\to 0} 2\check{\rho}(s)n = 1$. Therefore any point $x \in S_\nu$ satisfies $\lim_{s\to 0} 2\check{\rho}(s)x = L_\nu$. As the L_ν are the fixed points of the $\mathbb{G}_m$-action via $2\check{\rho}$, we see that

$$S_\nu \;=\; \{x \in \mathcal{G}\mathrm{r} \mid \lim_{s\to 0} 2\check{\rho}(s)x = L_\nu\}\,. \tag{3.5}$$

Hence, if $x \in S_\nu \cap \mathcal{G}\mathrm{r}^\lambda$ then, because $\mathcal{G}\mathrm{r}^\lambda$ is T-invariant, we see that $L_\nu \in \overline{\mathcal{G}\mathrm{r}^\lambda}$. Thus, $S_\nu \cap \mathcal{G}\mathrm{r}^\lambda$ is non-empty precisely when $L_\nu \in \overline{\mathcal{G}\mathrm{r}^\lambda}$. Let now ν be such that $L_\nu \in \overline{\mathcal{G}\mathrm{r}^\lambda}$. We begin with two extreme cases:

$$S_\nu \cap \overline{\mathcal{G}\mathrm{r}^\nu} = \; N_{\mathcal{O}} \cdot L_\nu = \begin{cases} I \cdot L_\nu & \text{if } \nu \text{ is dominant} \\ \{L_\nu\} & \text{if } \nu \text{ is anti-dominant} \end{cases} \tag{3.6}$$

We see this as follows. We first observe that $N_{\mathcal{K}} = N_{\mathcal{O}} \cdot (N_{\mathcal{K}} \cap K_-)$. Then we can write

$$S_\nu \cap \overline{\mathcal{G}\mathrm{r}^\nu} = \; N_{\mathcal{O}} \cdot (N_{\mathcal{K}} \cap K_-) \cdot L_\nu \;\cap\; \overline{\mathcal{G}\mathrm{r}^\nu} \;=\; N_{\mathcal{O}} \cdot \left((N_{\mathcal{K}} \cap K_-) \cdot \nu \cap \overline{\mathcal{G}\mathrm{r}^\nu} \right)\,. \tag{3.7}$$

But now $(N_{\mathcal{K}} \cap K_-) \cdot L_\nu \subset K_- \cdot L_\nu$ and by (2.6) we know that $G^-_{\mathcal{O}} \cdot L_\nu \cap \overline{\mathcal{G}\mathrm{r}^\lambda} = G \cdot L_\nu$ and because $K_- \cdot L_\nu$ is the fiber of the projection $G^-_{\mathcal{O}} \cdot L_\nu \to G \cdot L_\lambda$, we get $K_- \cdot \nu \cap \overline{\mathcal{G}\mathrm{r}^\lambda} = L_\nu$. Thus we have proved the first equality in (3.6). If ν is antidominant, then $N_{\mathcal{O}}$ stabilizes L_ν. If ν is dominant, then $N^-_{\mathcal{O}}$ stabilizes L_ν and then $I \cdot L_\nu = B_{\mathcal{O}} \cdot N^-_{\mathcal{O}} \cdot L_\nu = B_{\mathcal{O}} \cdot L_\nu = N_{\mathcal{O}} \cdot L_\nu$.

From (3.6) we conclude that the theorem holds in the extreme cases when $\nu = \lambda$ or $\nu = w_0 \cdot \lambda$, where w_0 is the longest element in the Weyl group. We consider an arbitrary ν such that $L_\nu \in \overline{\mathcal{G}\mathrm{r}^\lambda}$, $\nu > w_0 \cdot \lambda$. It suffices to prove the dimension estimate for closures, i.e., that $\overline{S_\nu} \cap \overline{\mathcal{G}\mathrm{r}^\lambda}$ is of pure dimension $\rho(\nu+\lambda)$.

Let us consider an irreducible component C of $\overline{S_\nu} \cap \overline{\mathcal{G}\mathrm{r}^\lambda}$. We construct a path $\lambda = \nu_0, \nu_1, \ldots, \nu_d = \nu$, where $\nu_k - \nu_{k+1} = \alpha_{i_k}$ for a simple positive coroot α_{i_k}, and a sequence C_k of irreducible components of $\overline{S_{\nu_k}} \cap \overline{\mathcal{G}\mathrm{r}^\lambda}$ containing C such that C_{k+1} is of codimension at most one in C_k, as follows. Let $d = \rho(\lambda - \nu)$ and let $\nu_d = \nu$ and $C_d = C$. Since $\overline{S_{\nu_d}} \subset \overline{S_{\nu_{d-1}}}$ some irreducible component C_{d-1} of $\overline{S_{\nu_{d-1}}} \cap \overline{\mathcal{G}\mathrm{r}^\lambda}$ contains C_d. Proceeding in this way we get $C_0 = \overline{\mathcal{G}\mathrm{r}^\lambda}$. Because of 3.1, $\overline{S_{\nu_k}} - S_{\nu_k}$ is given by a hyperplane section H_k on $\overline{S_{\nu_k}}$, so C_{k+1} is one of the irreducible components of $H_k \cap C_k$. Therefore, the codimension of C_{k+1} in C_k is at most one and we conclude:

$$\dim C \geq \dim \overline{\mathcal{G}\mathrm{r}^\lambda} - d = \rho(\nu + \lambda)\,. \tag{3.8}$$

Exactly by the same process as above we construct a path $\nu = \nu_0, \nu_1, \ldots, \nu_{d'} = w_0 \cdot \lambda$ where again $\nu_k - \nu_{k+1} = \alpha_{i_k}$, for a simple positive coroot α_{i_k}, starting now with $C_0 = C$ and ending with $C_{d'} = \overline{S_{w_0\cdot\lambda}} \cap \overline{\mathcal{G}\mathrm{r}^\lambda} = \{L_{w_0\cdot\lambda}\}$. Now $d' = \rho(\nu - w_0 \cdot \lambda)$ and hence $0 = \dim(C_{d'}) \geq \dim C - \rho(\nu - w_0 \cdot \lambda)$. Thus

$$\dim C \leq \rho(\nu - w_0 \cdot \lambda) = \rho(\nu + \lambda)\,. \tag{3.9}$$

□

The corollary below will be used to construct the convolution operation on perverse sheaves in the next section.

3.4. **Corollary.** *For any dominant $\lambda \in X_*(T)$ and any T-invariant closed subset $X \subset \overline{\mathcal{G}r^\lambda}$ we have $\dim(X) \leq \max_{L_\nu \in X^T} \rho(\lambda + \nu)$, where X^T stands for the set of T–fixed points of X.*

Proof. From the description (3.5) we see that $X \cap S_\nu$ is non-empty precisely when $L_\nu \in X$. As

$$X \;=\; \cup_{L_\nu \in X^T} X \cap S_\nu \subset \cup_{L_\nu \in X^T} \overline{\mathcal{G}\mathrm{r}^\lambda} \cap S_\nu\,, \tag{3.10}$$

we get our conclusion by appealing to the previous theorem. □

Let us write $\mathrm{Mod}_{\Bbbk}$ for the category of finitely generated $\Bbbk$-modules.

3.5. **Theorem.** *For all $\mathcal{A} \in \mathrm{P}_{G_\mathcal{O}}(\mathcal{G}r, \Bbbk)$ we have a canonical isomorphism*

$$\mathrm{H}^k_c(S_\nu, \mathcal{A}) \;\xrightarrow{\sim}\; \mathrm{H}^k_{T_\nu}(\mathcal{G}r, \mathcal{A}) \tag{3.11}$$

and both sides vanish for $k \neq 2\rho(\nu)$.

In particular, the functors $F_\nu : \mathrm{P}_{G_\mathcal{O}}(\mathcal{G}r, \Bbbk) \to \mathrm{Mod}_{\Bbbk}$, defined by $F_\nu \stackrel{def}{=} \mathrm{H}^{2\rho(\nu)}_c(S_\nu, -) = \mathrm{H}^{2\rho(\nu)}_{T_\nu}(\mathcal{G}r, -)$, are exact.

Proof. Let $\mathcal{A} \in \mathrm{P}_{G_\mathcal{O}}(\mathcal{G}\mathrm{r}, \Bbbk)$. For any dominant η the restriction $\mathcal{A}\big|\mathcal{G}\mathrm{r}^\eta$ lies, as a complex of sheaves, in degrees $\leq -\dim(\mathcal{G}\mathrm{r}^\eta) = -2\rho(\eta)$, i.e., $\mathcal{A}\big|\mathcal{G}\mathrm{r}^\eta \in D^{\leq -2\rho(\eta)}(\mathcal{G}\mathrm{r}^\eta, \Bbbk)$. From the dimension estimates 3.2 and fact that $\mathrm{H}^k_c(S_\nu \cap \mathcal{G}\mathrm{r}^\eta, \Bbbk) = 0$, for $k > 2\dim(S_\nu \cap \mathcal{G}\mathrm{r}^\eta) = 2\rho(\nu + \eta)$ we conclude:

$$\mathrm{H}^k_c(S_\nu \cap \mathcal{G}\mathrm{r}^\eta, \mathcal{A}) \;=\; 0 \quad \text{if } k > 2\rho(\nu)\,. \tag{3.12}$$

A straightforward spectral sequence argument, filtering $\mathcal{G}\mathrm{r}$ by $\overline{\mathcal{G}\mathrm{r}^\eta}$, implies that $\mathrm{H}_c^*(S_\nu,\mathcal{A})$ can be expressed in terms of $\mathrm{H}_c^*(S_\nu\cap\mathcal{G}\mathrm{r}^\nu,\mathcal{A})$ and this implies the first of the statements below:

$$
\begin{aligned}
\mathrm{H}_c^k(S_\nu,\mathcal{A}) &= 0 \quad \text{if } k>2\rho(\nu)\\
\mathrm{H}_{T_\nu}^k(\mathcal{G}\mathrm{r},\mathcal{A}) &= 0 \quad \text{if } k<2\rho(\nu)\,.
\end{aligned}
\tag{3.13}
$$

The proof for the second statement is completely analogous.

It remains to prove (3.11). Recall that we have a $\mathbb{G}_m$-action on $\mathcal{G}\mathrm{r}$ via the cocharacter $2\check\rho$ whose fixed points are the points L_ν, $\nu\in X_*(T)$, and that

$$S_\nu = \{x\in\mathcal{G}\mathrm{r} \mid \lim_{s\to 0} 2\check\rho(s)x = L_\nu\} \tag{3.14}$$

$$T_\nu = \{x\in\mathcal{G}\mathrm{r} \mid \lim_{s\to \infty} 2\check\rho(s)x = L_\nu\}\,. \tag{3.15}$$

The statement (3.11) now follows from theorem 1 in [Br].

□

We will denote by $F:\ \mathrm{P}_{G_\mathcal{O}}(\mathcal{G}\mathrm{r},\Bbbk)\to \mathrm{Mod}_\Bbbk$ the sum of the functors $F_\nu,\ \nu\in\ X_*(T)$.

3.6. **Theorem.** *We have a natural equivalence of functors*

$$\mathrm{H}^* \ \cong\ F\ =\ \bigoplus_{\nu\in X_*(T)} \mathrm{H}_c^{2\rho(\nu)}(S_\nu,-)\ :\ \mathrm{P}_{G_\mathcal{O}}(\mathcal{G}r,\Bbbk)\to\mathrm{Mod}_\Bbbk\,.$$

Furthermore, this equivalence and the functors F_ν are independent of the choice of the pair $T\subset B$.

Proof. The cohomology functor H^* has two filtrations indexed by $X_*(T)$. One is given by kernels of of the functors $\mathrm{H}^*\to\mathrm{H}_c^*(\bar S_\nu,-)$ and the other by images of the functors $\mathrm{H}^*_{\bar T_\nu}(\mathcal{G}\mathrm{r},-)\to\mathrm{H}^*$. The vanishing statement in 3.5 implies that these filtrations are complementary. More precisely, in degree $2\rho(\nu)$ we get $\mathrm{H}^{2\rho(\nu)}_{\bar T_\nu}(\mathcal{G}\mathrm{r},-)=\mathrm{H}^{2\rho(\nu)}_{T_\nu}(\mathcal{G}\mathrm{r},-)$, $\mathrm{H}_c^{2\rho(\nu)}(\bar S_\nu,-)=\mathrm{H}_c^{2\rho(\nu)}(S_\nu,-)$, and the composition of the functors $\mathrm{H}^{2\rho(\nu)}_{T_\nu}(\mathcal{G}\mathrm{r},-)\to\mathrm{H}^{2\rho(\nu)}\to\mathrm{H}_c^{2\rho(\nu)}(S_\nu,-)$ is the canonical equivalence in 3.5. Hence, the two filtrations of H^* split each other and provide the desired natural equivalence.

It remains to prove the independence of the equivalence and the functors F_ν of the choice of $T\subset B$. Let us fix a reference $T_0\subset B_0$ and a $\nu\in X_*(T_0)$ which gives us the $S_\nu^0=(N_0)_\mathcal{K}\cdot\nu$. The choice of pairs $T\subset B$ is parametrized by the variety G/T_0. Note that there is a canonical isomorphism between T and T_0; they are both canonically isomorphic to the "universal" Cartan $B_0/N_0=B/N$. Consider the following diagram

$$
\begin{array}{ccccc}
\mathcal{G}\mathrm{r} & \xleftarrow{\ p\ } & \mathcal{G}\mathrm{r}\times G/T_0 & \xleftarrow{\ j\ } & S\\
 & & {\scriptstyle q}\big\downarrow & & {\scriptstyle r}\big\downarrow\\
 & & G/T_0 & = & G/T_0\,.
\end{array}
\tag{3.16}
$$

Here p,q,r are projections and $S=\{(x,gT_0)\in\mathcal{G}\mathrm{r}\times G/T_0\mid x\in gS_\nu\}$. For a point in G/T_0, i.e., for a choice of $T\subset B$ the fiber of r is precisely the set S_ν of the pair. Now, for any $\mathcal{A}\in\mathrm{P}_{G_\mathcal{O}}(\mathcal{G}\mathrm{r},\Bbbk)$ the local system $Rq_*j_!j^*p^*\mathcal{A}$ is a sublocal system of $Rq_*p^*\mathcal{A}$.

As the latter local system is trivial, so is the former and hence the functors F_ν are independent of the choice of $T \subset B$. $\square$

3.7. **Corollary.** *The global cohomology functor* $\mathrm{H}^* = F : \mathrm{P}_{G_\mathcal{O}}(\mathcal{G}r, \Bbbk) \to \mathrm{Mod}_\Bbbk$ *is faithful and exact.*

Proof. The exactness follows from 3.5 and 3.6. If $\mathcal{A} \in \mathrm{P}_{G_\mathcal{O}}(\mathcal{G}\mathrm{r}, \Bbbk)$ is non-zero then there exists an orbit $\mathcal{G}\mathrm{r}^\lambda$ which is open in the support of $\mathcal{A}$. If we choose λ dominant then T_λ is a point in $\mathcal{G}\mathrm{r}^\lambda$ and we see that $F_\lambda(\mathcal{A}) \neq 0$. As H^* does not annihilate non-zero objects it is faithful. $\square$

3.8. **Remark.** *The decompositions for N and its opposite unipotent subgroup $\bar{N}$ are explicitly related by a canonical identification $\mathrm{H}^k_{S_\nu}(\mathcal{G}r, \mathcal{A}) \cong \mathrm{H}^k_{T_{w_0 \cdot \nu}}(\mathcal{G}r, \mathcal{A})$, given by the action of any representative of w_0, the longest element in the Weyl group.*

We have the following criterion for a sheaf to be perverse

3.9. **Lemma.** *A sheaf $\mathcal{A} \in \mathrm{D}_{G_\mathcal{O}}(\mathcal{G}r, \Bbbk)$ is perverse if and only if for all $\nu \in X_*(T)$ the cohomology group $\mathrm{H}^*_c(S_\nu, \mathcal{A})$ is zero except possibly in degree $2\rho(\nu)$.*

Proof. By 3.5 and 3.6 and an easy spectral sequence argument one concludes that $\mathrm{H}^{2\rho(\nu)}_c(S_\nu, {}^p\mathrm{H}^k(\mathcal{A})) = \mathrm{H}^{2\rho(\nu)+k}_c(S_\nu, \mathcal{A})$. This forces $\mathcal{A}$ to be perverse. $\square$

Finally, we use the results of this section to give a rather explicit geometric description of the cohomology of the standard sheaves $\mathcal{I}_!(\lambda, \Bbbk)$ and $\mathcal{I}_*(\lambda, \Bbbk)$.

3.10. **Proposition.** *There are canonical identifications*

$$F_\nu[\mathcal{I}_!(\lambda, \Bbbk)] \;\cong\; \Bbbk[Irr(\overline{\mathcal{G}r_\lambda} \cap S_\nu)] \;\cong\; F_\nu[\mathcal{I}_*(\lambda, \Bbbk)];$$

here $\Bbbk[Irr(\overline{\mathcal{G}r_\lambda} \cap S_\nu)]$ stands for the free $\Bbbk$-module generated by the irreducible components of $\overline{\mathcal{G}r_\lambda} \cap S_\nu$.

Proof. We will give the argument for $\mathcal{I}_!(\lambda, \Bbbk)$. The argument for $\mathcal{I}_*(\lambda, \Bbbk)$ is completely analogous. We proceed precisely the same way as in the beginning of the proof of 3.5. Let us write $\mathcal{A} = \mathcal{I}_!(\lambda, \Bbbk)$. Consider an orbit $\mathcal{G}\mathrm{r}^\eta$ in the boundary of $\mathcal{G}\mathrm{r}^\lambda$. Then $\mathcal{A}\big|\mathcal{G}\mathrm{r}^\eta \in D^{\leq -\dim(\mathcal{G}\mathrm{r}^\eta)-2}(\mathcal{G}\mathrm{r}^\eta, \Bbbk)$. The estimate 3.12 implies that $\mathrm{H}^k_c(S_\nu \cap \mathcal{G}\mathrm{r}^\eta, \mathcal{A}) = 0$ if $k > 2\rho(\nu) - 2$. Therefore, we conclude by using the spectral sequence associated to the filtration of $\mathcal{G}\mathrm{r}$ by $\overline{\mathcal{G}^\eta}$ that $\mathrm{H}^{2\rho(\nu)}_c(S_\nu, \mathcal{A}) \cong \mathrm{H}^{2\rho(\nu)}_c(S_\nu \cap \mathcal{G}\mathrm{r}^\lambda, \mathcal{A})$. Finally,

$$\mathrm{H}^{2\rho(\nu)}_c(S_\nu \cap \mathcal{G}\mathrm{r}^\lambda, \mathcal{A}) = \mathrm{H}^{2\rho(\nu+\lambda)}_c(S_\nu \cap \mathcal{G}\mathrm{r}^\lambda, \Bbbk) = \mathrm{H}^{2\dim(S_\nu \cap \mathcal{G}\mathrm{r}^\lambda)}_c(S_\nu \cap \mathcal{G}\mathrm{r}^\lambda, \Bbbk)\,. \tag{3.17}$$

As the last cohomology group is the top cohomology group, it is a free $\Bbbk$-module with basis $Irr(\overline{\mathcal{G}\mathrm{r}}^\lambda \cap S_\nu)$. $\square$

4. Convolution product

In this section we will put a tensor category structure on $\mathrm{P}_{G_\mathcal{O}}(\mathcal{G}\mathrm{r}, \Bbbk)$ via the convolution product. In some of our constructions in this section and the next one we are lead to sheaves with infinite dimensional support. The fact that it is legitimate to work with such objects is explained in section 2.2 of [Na].

Consider the following diagram of maps

$$\mathcal{G}r \times \mathcal{G}r \xleftarrow{p} G_{\mathcal{K}} \times \mathcal{G}r \xrightarrow{q} G_{\mathcal{K}} \times_{G_{\mathcal{O}}} \mathcal{G}r \xrightarrow{m} \mathcal{G}r\,. \tag{4.1}$$

Here $G_{\mathcal{K}} \times_{G_{\mathcal{O}}} \mathcal{G}r$ denotes the quotient of $G_{\mathcal{K}} \times \mathcal{G}r$ by $G_{\mathcal{O}}$ where the action is given on the $G_{\mathcal{K}}$-factor via right multiplication by an inverse and on the $\mathcal{G}r$-factor by left multiplication. The p and q are projection maps and m is the multiplication map. We define the convolution product

$$\mathcal{A}_1 * \mathcal{A}_2 \;=\; Rm_*\tilde{\mathcal{A}} \qquad \text{where} \;\; q^*\tilde{\mathcal{A}} = \; p^*({}^p\mathrm{H}^0(\mathcal{A}_1 \boxtimes \mathcal{A}_2))\,. \tag{4.2}$$

To justify this definition, we note that the sheaf $p^*({}^p\mathrm{H}^0(\mathcal{A}_1 \boxtimes \mathcal{A}_2))$ on $G_{\mathcal{K}} \times \mathcal{G}r$ is $G_{\mathcal{O}} \times G_{\mathcal{O}}$-equivariant with the first $G_{\mathcal{O}}$ acting on the left and the second $G_{\mathcal{O}}$ acting on the $G_{\mathcal{K}}$-factor via right multiplication by an inverse and on the $\mathcal{G}r$-factor by left multiplication. As the second $G_{\mathcal{O}}$-action is free, we see that the unique $\tilde{\mathcal{A}}$ in (4.1) exists.

4.1. **Proposition.** *The convolution product $\mathcal{A}_1 * \mathcal{A}_2$ of two perverse sheaves is perverse.*

To prove this, let us introduce the notion of a stratified semi-small map. To this end, let us consider two complex stratified spaces $(Y, \mathcal{T})$ and $(X, \mathcal{S})$ and a map $f : Y \to X$. We assume that the two stratifications are locally trivial with connected strata and that f is a stratified with respect to the stratifications $\mathcal{T}$ and $\mathcal{S}$, i.e., that for any $T \in \mathcal{T}$ the image $f(T)$ is a union of strata in $\mathcal{S}$ and for any $S \in \mathcal{S}$ the map $f\big|f^{-1}(S) : f^{-1}(S) \to S$ is locally trivial in the stratified sense. We say that f is a stratified semi-small map if

(4.3)

a) for any $T \in \mathcal{T}$ the map $f\big|\bar{T}$ is proper

b) for any $T \in \mathcal{T}$ and any $S \in \mathcal{S}$ such that $S \subset f(\bar{T})$ we have
$$\dim(f^{-1}(x) \cap \bar{T}) \le \frac{1}{2}(\dim f(\bar{T}) - \dim S)$$
for any (and thus all) $x \in S$.

Let us also introduce the notion of a small stratified map. We say that f is a small stratified map if there exists a (non-trivial) open dense stratified subset W of Y such that

(4.4)

a) for any $T \in \mathcal{T}$ the map $f\big|\overline{T}$ is proper

b) the map $f\big|W : W \to f(W)$ is finite and $W = f^{-1}(f(W))$

c) for any $T \in \mathcal{T}$ and any $S \in \mathcal{S}$ such that $S \subset f(\overline{T}) - f(W)$
we have $\dim(f^{-1}(x) \cap \overline{T}) < \frac{1}{2}(\dim f(\overline{T}) - \dim S)$
for any (and thus all) $x \in S$.

The result below follows directly from dimension counting:

4.2. **Lemma.** *If f is a semi-small stratified map then $Rf_*\mathcal{A} \in \mathrm{P}_{\mathcal{S}}(X, \Bbbk)$ for all $\mathcal{A} \in \mathrm{P}_{\mathcal{T}}(Y, \Bbbk)$. If f is a small stratified map then, with any W as above, and any $\mathcal{A} \in \mathrm{P}_{\mathcal{T}}(W, \Bbbk)$, we have $Rf_* j_{!*}\mathcal{A} = \tilde{j}_{!*} f_*\mathcal{A}$, where $j : W \hookrightarrow Y$ and $\tilde{j} : f(W) \hookrightarrow X$ denote the two inclusions.*

We apply the above considerations, in the semi-small case, to our situation. We take $Y = G_{\mathcal{K}} \times_{G_{\mathcal{O}}} \mathcal{G}\mathrm{r}$ and choose $\mathcal{T}$ to be the stratification whose strata are $\tilde{\mathcal{G}\mathrm{r}}^{\lambda,\mu} = p^{-1}(\mathcal{G}\mathrm{r}^{\lambda}) \times_{G_{\mathcal{O}}} \mathcal{G}\mathrm{r}^{\mu}$, for $\lambda, \mu \in X_*(T)$. We also let $X = \mathcal{G}\mathrm{r}$, $\mathcal{S}$ the stratification by $G_{\mathcal{O}}$-orbits, and choose $f = m$. Note that the sheaf $\tilde{\mathcal{A}}$ is constructible with respect to the stratification $\mathcal{T}$. To be able to apply 4.2 and conclude the proof of 4.1, we appeal to the following

4.3. **Lemma.** *The multiplication map $G_{\mathcal{K}} \times_{G_{\mathcal{O}}} \mathcal{G}r \xrightarrow{m} \mathcal{G}r$ is a stratified semi-small map with respect to the stratifications above.*

Proof. We need to check that for any $G_{\mathcal{O}}$-orbit $\mathcal{G}\mathrm{r}^{\nu}$ in $\overline{\mathcal{G}\mathrm{r}^{\lambda+\mu}}$, the dimension of the fiber $m^{-1}L_{\nu} \cap \tilde{\mathcal{G}\mathrm{r}}^{\lambda,\mu}$ of $m : \tilde{\mathcal{G}\mathrm{r}}^{\lambda,\mu} \to \overline{\mathcal{G}\mathrm{r}^{\lambda+\mu}}$ at L_{ν}, is not more than $\frac{1}{2}\mathrm{codim}_{\overline{\mathcal{G}\mathrm{r}^{\lambda+\mu}}}\mathcal{G}\mathrm{r}_{\nu}$. We can assume that ν is anti-dominant since $\mathcal{G}\mathrm{r}^{w\cdot\eta} = \mathcal{G}\mathrm{r}^{\eta},\ w \in W$. Since for any dominant η, $\dim \mathcal{G}\mathrm{r}^{\eta} = 2\rho(\eta)$, the codimension in question is:

$$\mathrm{codim}_{\overline{\mathcal{G}\mathrm{r}^{\lambda+\mu}}}\mathcal{G}\mathrm{r}^{\nu} = 2\rho(\lambda+\mu) - 2\rho(w_0\cdot\nu) = 2\rho(\lambda+\mu+\nu).$$

Therefore, we need to show that

$$\dim(m^{-1}L_{\nu} \cap \tilde{\mathcal{G}\mathrm{r}}^{\lambda,\mu}) \leq \ \rho(\lambda+\mu+\nu)\,. \tag{4.5}$$

Let p be the projection $G_{\mathcal{K}} \times_{G_{\mathcal{O}}} \mathcal{G}\mathrm{r} \to \mathcal{G}\mathrm{r}$ given by $(g, hG_{\mathcal{O}}) \mapsto gG_{\mathcal{O}}$, and consider the isomorphism $(p, m) :\ G_{\mathcal{K}} \times_{G_{\mathcal{O}}} \mathcal{G}\mathrm{r} \ \cong\ \mathcal{G}\mathrm{r} \times \mathcal{G}\mathrm{r}$. The mapping (p, m) carries the fiber $m^{-1}L_{\nu}$ to $\mathcal{G}\mathrm{r} \times L_{\nu}$. The set $\dim(p(m^{-1}L_{\nu} \cap \overline{\tilde{\mathcal{G}\mathrm{r}}^{\lambda,\mu}}))$ is T-invariant, and hence we can apply corollary 3.4 to compute its dimension. To do so, we need to find the T-fixed points in $p(m^{-1}L_{\nu} \cap \overline{\tilde{\mathcal{G}\mathrm{r}}^{\lambda,\mu}}) \ \subset\ \overline{\mathcal{G}\mathrm{r}^{\lambda}}$. The T-fixed points in $m^{-1}L_{\nu} \cap \tilde{\mathcal{G}\mathrm{r}}^{\lambda,\mu}$ are precisely the points $(z^{\phi}, z^{\psi}G_{\mathcal{O}})$ such that ϕ and ψ are weights of $L(\lambda)$ and $L(\mu)$ and $\phi + \psi =\ \nu$. Hence, the set T-fixed points in $m^{-1}L_{\nu} \cap \overline{\tilde{\mathcal{G}\mathrm{r}}^{\lambda,\mu}}$ consists of the points of the form $(z^{\phi}, z^{\psi}G_{\mathcal{O}})$ with $\phi + \psi =\ \nu$ and ϕ and ψ weights of irreducible representations $L(\lambda')$ and $L(\mu')$ for some dominant λ', μ' such that $\lambda' \leq \lambda,\ \mu' \leq \mu$. For ϕ, ψ, μ' as above, we have

$$\rho(\lambda+\phi) \ \leq\ \rho(\lambda+\phi) + \rho(\psi+\mu') =\ \rho(\lambda+\nu+\mu') \leq\ \rho(\lambda+\nu+\mu)\,.$$

Therefore,

$$\dim(p(m^{-1}L_{\nu} \cap \overline{\tilde{\mathcal{G}\mathrm{r}}^{\lambda,\mu}})) \ \leq \max_{L_{\phi} \in p(m^{-1}L_{\nu} \cap \overline{\tilde{\mathcal{G}\mathrm{r}}^{\lambda,\mu}})^{T}} (\rho(\lambda+\phi)) \leq \rho(\lambda+\nu+\mu)\,. \tag{4.6}$$

This implies (4.5) and concludes the proof.

□

In completely analogy with (4.2), we can define directly the convolution product of three sheaves, i.e., to $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ we can associate a perverse sheaf $\mathcal{A}_1 * \mathcal{A}_2 * \mathcal{A}_3$. Furthermore, we get canonical isomorphisms $\mathcal{A}_1 * \mathcal{A}_2 * \mathcal{A}_3 \cong (\mathcal{A}_1 * \mathcal{A}_2) * \mathcal{A}_3$ and $\mathcal{A}_1 * \mathcal{A}_2 * \mathcal{A}_3 \cong \mathcal{A}_1 * (\mathcal{A}_2 * \mathcal{A}_3)$. This yields a functorial isomorphism $(\mathcal{A}_1 * \mathcal{A}_2) * \mathcal{A}_3 \cong \mathcal{A}_1 * (\mathcal{A}_2 * \mathcal{A}_3)$ and hence we get

4.4. **Proposition.** *The abelian category* $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}r,\Bbbk)$*, equipped with the convolution product* (4.2)*, has the structure of an associative tensor category (i.e., a category with a tensor functor supplied with an associativity constraint* [DM]*).*

If $\Bbbk$ is a field, or, more generally, if one of the factors $\mathrm{H}^*(\mathcal{G}\mathrm{r},\mathcal{A}_i)$ is flat the outer tensor product $\mathcal{A}_1 \boxtimes \mathcal{A}_2$ is perverse. One sees this by applying 3.9 to the Grassmannian $\mathcal{G}\mathrm{r}\times\mathcal{G}\mathrm{r}$ of $G\times G$.

5. **The commutativity constraint and the fusion product**

In this section we show that the convolution product defined in the last section can be viewed as a "fusion" product. This interpretation allows one to provide the convolution product on $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r},\Bbbk)$ with a commutativity constraint, making $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r},\Bbbk)$ into an associative, commutative tensor category. The exposition follows very closely that in [MiV2]. The idea of interpreting the convolution product as a fusion product and obtaining the commutativity constraint in this fashion is due to Drinfeld.

Let X be a smooth complex algebraic curve. For a closed point $x \in X$ we write $\mathcal{O}_x$ for the completion of the local ring at x and $\mathcal{K}_x$ for its fraction field. Furthermore, for a $\mathbb{C}$-algebra R we write $X_R = X \times \mathrm{Spec}(R)$, and $X_R^* = (X - \{x\}) \times \mathrm{Spec}(R)$. Using the results of [BL1, BL2, LS] we can now view the Grassmannian $\mathcal{G}\mathrm{r}_x = G_{\mathcal{K}_x}/G_{\mathcal{O}_x}$ in the following manner. It is the ind-scheme which represents the functor from $\mathbb{C}$-algebras to sets :

$$R \mapsto \{\mathcal{F} \text{ a } G\text{-torsor on } X_R,\ \nu : G \times X_R^* \to \mathcal{F}\big|X_R^* \text{ a trivialization on } X_R^*\ \}\,.$$

Here the pairs $(\mathcal{F},\nu)$ are to be taken up to isomorphism.

Following [BD] we globalize this construction and at the same time work over several copies of the curve. Denote the n fold product by $X^n = X \times \cdots \times X$ and consider the functor

$$R \mapsto \left\{ \begin{array}{l} (x_1,\dots,x_n) \in X^n(R), \quad \mathcal{F} \text{ a } G\text{-torsor on } X_R\,, \\ \nu_{(x_1,\dots,x_n)} \text{ a trivialization of } \mathcal{F} \text{ on } X_R - \cup x_i \end{array} \right\}. \tag{5.1}$$

Here we think of the points $x_i : \mathrm{Spec}(R) \to X$ as subschemes of X_R by taking their graphs. This functor is represented by an ind-scheme $\mathcal{G}\mathrm{r}_{X^n}$. Of course $\mathcal{G}\mathrm{r}_{X^n}$ is an ind-scheme over X^n and its fiber over the point $(x_1,\dots,x_n)$ is simply $\prod_{i=1}^k \mathcal{G}\mathrm{r}_{y_i}$, where $\{y_1,\dots,y_k\} = \{x_1,\dots,x_n\}$, with all the y_i distinct. We write $\mathcal{G}\mathrm{r}_{X^1} = \mathcal{G}\mathrm{r}_X$.

We will now extend the diagram of maps (4.1), which was used to define the convolution product, to the global situation, i.e., to a diagram of ind-schemes over X^2:

$$\mathcal{G}\mathrm{r}_X \times \mathcal{G}\mathrm{r}_X \xleftarrow{p} \widetilde{\mathcal{G}\mathrm{r}_X \times \mathcal{G}\mathrm{r}_X} \xrightarrow{q} \mathcal{G}\mathrm{r}_X \tilde{\times} \mathcal{G}\mathrm{r}_X \xrightarrow{m} \mathcal{G}\mathrm{r}_{X^2}\,. \tag{5.2}$$

Here, $\widetilde{\mathcal{G}\mathrm{r}_X \times \mathcal{G}\mathrm{r}_X}$ denotes the ind-scheme representing the functor

$$R \mapsto \left\{ \begin{array}{l} (x_1,x_2) \in X^2(R);\ \mathcal{F}_1,\mathcal{F}_2 \text{ } G\text{-torsors on } X_R;\ \nu_i \text{ a trivialization of} \\ \mathcal{F}_i \text{ on } X_R - x_i, \text{ for } i=1,2;\ \ \mu_1 \text{ a trivialization of } \mathcal{F}_1 \text{ on } \widehat{(X_R)}_{x_2} \end{array} \right\}, \tag{5.3}$$

where $\widehat{(X_R)}_{x_2}$ denotes the formal neighborhood of x_2 in X_R. The "twisted product" $\mathcal{G}r_X \tilde{\times} \mathcal{G}r_X$ is the ind-scheme representing the functor

$$R \mapsto \left\{ \begin{array}{l} (x_1, x_2) \in X^2(R);\ \mathcal{F}_1, \mathcal{F}\ G\text{-torsors on } X_R;\ \nu_1 \text{ a trivialization} \\ \text{of } \mathcal{F}_1 \text{ on } X_R - x_1;\ \eta : \mathcal{F}_1\big|(X_R - x_2) \xrightarrow{\simeq} \mathcal{F}\big|(X_R - x_2) \end{array} \right\}. \tag{5.4}$$

It remains to describe the morphisms p, q, and m in (5.2). Because all the spaces in (5.2) are ind-schemes over X^2, and all the functors involve the choice of the same point $(x_1, x_2) \in X^2(R)$, we omit it in the formulas below. The morphism p simply forgets the choice of μ_1, the morphism q is given by the natural transformation

$$(\mathcal{F}_1, \nu_1, \mu_1; \mathcal{F}_2, \nu_2) \mapsto (\mathcal{F}_1, \nu_1, \mathcal{F}, \eta),$$

where $\mathcal{F}$ is the G-torsor gotten by gluing $\mathcal{F}_1$ on $X_R - x_2$ and $\mathcal{F}_2$ on $\widehat{(X_R)}_{x_2}$ using the isomorphism induced by $\nu_2 \circ \mu_1^{-1}$ between $\mathcal{F}_1$ and $\mathcal{F}_2$ on $(X_R - x_2) \cap \widehat{(X_R)}_{x_2}$. The morphism m is given by the natural transformation

$$(\mathcal{F}_1, \nu_1, \mathcal{F}, \eta) \mapsto (\mathcal{F}, \nu)\,,$$

where $\nu = (\eta \circ \nu_1)\big|(X_R - x_1 - x_2)$.

The global analogue of $G_{\mathcal{O}}$ is the group-scheme $G_{X^n,\mathcal{O}}$ which represents the functor

$$R \mapsto \left\{ \begin{array}{l} (x_1, \dots, x_n) \in X^n(R),\quad \mathcal{F} \text{ the trivial } G\text{-torsor on } X_R\,, \\ \mu_{(x_1,\dots,x_n)} \text{ a trivialization of } \mathcal{F} \text{ on } \widehat{(X_R)}_{(x_1 \cup \cdots \cup x_n)} \end{array} \right\}. \tag{5.5}$$

Proceeding as in section 4 we define the convolution product of $\mathcal{B}_1, \mathcal{B}_2 \in \mathrm{P}_{G_{X,\mathcal{O}}}(\mathcal{G}r_X, \Bbbk)$ by the formula

$$\mathcal{B}_1 *_X \mathcal{B}_2 \;=\; Rm_* \tilde{\mathcal{B}} \qquad \text{where} \quad q^* \tilde{\mathcal{B}} = p^*(\mathcal{B}_1 \boxtimes \mathcal{B}_2)\,. \tag{5.6}$$

The existence the sheaf $\tilde{\mathcal{B}}$ can be justified in the same manner as in the definition of the convolution product. Furthermore, the map m is a stratified small map – regardless of the stratification on X. To see this, let us denote by $\Delta \subset X^2$ the diagonal and set $U = X^2 - \Delta$. Then we can take, in definition 4.4, as W the locus of points lying over U. That m is small now follows as m is an isomorphism over U and over points of Δ the map m coincides with its analogue in section 4 which is semi-small by proposition 4.3.

We will now construct the commutativity constraint. For simplicity we specialize to the case $X = \mathbb{A}^1$. The advantage is that we can once and for all choose a global coordinate. Then the choice of a global coordinate on $\mathbb{A}^1$, trivializes $\mathcal{G}r_X$ over X; let us write $\tau : \mathcal{G}r_X \to \mathcal{G}r$ for the projection. Let us denote $\tau^0 = \tau^*[1] : \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}r, \Bbbk) \to \mathrm{P}_{G_{X,\mathcal{O}}}(\mathcal{G}r_X, \Bbbk)$. By restricting $\mathcal{G}r_{X^{(2)}}$ to the diagonal $\Delta \cong X$ and to U, and observing that these restrictions are isomorphic to $\mathcal{G}r_X$ and to $(\mathcal{G}r_X \times \mathcal{G}r_X)\big|U$ respectively, we get the following diagram

$$\begin{array}{ccccc} \mathcal{G}r_X & \xrightarrow{\ i\ } & \mathcal{G}r_{X^2} & \xleftarrow{\ j\ } & (\mathcal{G}r_X \times \mathcal{G}r_X)\big|U \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ X & \longrightarrow & X^2 & \longleftarrow & U \end{array}\,. \tag{5.7}$$

For $\mathcal{A}_1, \mathcal{A}_2 \in \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}, \Bbbk)$ we have:

$$\begin{aligned} &\text{a)} \qquad \tau^0\mathcal{A}_1 *_X \tau^0\mathcal{A}_2 \;\cong\; j_{!*}\left((\tau^0\mathcal{A}_1 \boxtimes \tau^0\mathcal{A}_2)\big|U\right) \\ &\text{b)} \qquad \tau^0(\mathcal{A}_1 * \mathcal{A}_2) \;\cong\; i^0(\tau^0\mathcal{A}_1 *_X \tau^0\mathcal{A}_2)\,. \end{aligned} \tag{5.8}$$

Part a) follows from smallness of m and lemma 4.2, and part b) follows directly from definitions.

Utilizing the the statements above yields the following sequence of isomorphisms:

$$\begin{aligned} \tau^0(\mathcal{A}_1 * \mathcal{A}_2) &\cong i^0 j_{!*}\left((\tau^0\mathcal{A}_1 \boxtimes \tau^0\mathcal{A}_2)\big|U\right) \\ &\cong i^* j_{!*}((\tau^0\mathcal{A}_2 \boxtimes \tau^0\mathcal{A}_1)\big|U) \cong \tau^0(\mathcal{A}_2 * \mathcal{A}_1)\,. \end{aligned} \tag{5.9}$$

Specializing this isomorphism to (any) point on the diagonal yields a functorial isomorphism between $\mathcal{A}_1 * \mathcal{A}_2$ and $\mathcal{A}_2 * \mathcal{A}_1$. This gives us a commutativity constraint making $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}, \Bbbk)$ into a tensor category. In the next section we modify this commutativity constraint slightly. It will be the modified commutativity constraint that will be used in the rest of the paper.

5.1. **Remark.** *One can avoid having to specialize to the case $X = \mathbb{A}^1$ here, as well as in the next section. This can be done by dealing with all choices of a local coordinate at all points of the curve X. This gives rise to the* $\mathrm{Aut}(\mathcal{O})$*-torsor $\hat{X} \to X$. The functor $\tau^0 : \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}r, \Bbbk) \to \mathrm{P}_{G_{X,\mathcal{O}}}(\mathcal{G}r_X, \Bbbk)$ is constructed by noting that $\mathcal{G}r_X \to X$ is the fibration associated to the* $\mathrm{Aut}(\mathcal{O})$*-torsor $\hat{X} \to X$ and the* $\mathrm{Aut}(\mathcal{O})$*-action on $\mathcal{G}r$. By proposition 2.2, sheaves in $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}r, \Bbbk)$ are* $\mathrm{Aut}(\mathcal{O})$*-equivariant and hence we can transfer them to sheaves on $\mathcal{G}r_X$.*

6. Tensor functors

In this section we will to show that our functor

$$\mathrm{H}^* \;\cong\; F \;=\; \bigoplus_{\nu \in X_*(T)} \mathrm{H}_c^{2\rho(\nu)}(S_\nu, -) \;:\; \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}, \Bbbk) \to \mathrm{Mod}_{\Bbbk} \tag{6.1}$$

is a tensor functor. To this end let us write $\mathrm{Mod}_{\Bbbk}^{\epsilon}$ for the tensor category of finitely generated $\mathbb{Z}/2\mathbb{Z}$ - graded (super) modules over $\Bbbk$. Let us consider the global cohomology functor as a functor $\mathrm{H}^* : \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}, \Bbbk) \to \mathrm{Mod}_{\Bbbk}^{\epsilon}$; here we only keep track of the parity of the grading on global cohomology. Then:

6.1. **Lemma.** *The functor $\mathrm{H}^* : \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}r, \Bbbk) \to \mathrm{Mod}_{\Bbbk}^{\epsilon}$ is a tensor functor with respect to the commutativity constraint of the previous section.*

Proof. We use the interpretation of the convolution product as a fusion product, explained in the previous section. Let us write $r_2 : \mathcal{G}\mathrm{r}_X^2 \to X^2$ for the projection and again set $X = \mathbb{A}^1$. The lemma is an immediate consequence of the following statements:

(6.2a) $R(r_2)_*(\tau^0(\mathcal{A}_1) *_X \tau^0(\mathcal{A}_2))|U$ is the constant sheaf $\mathrm{H}^*(\mathcal{G}\mathrm{r}, \mathcal{A}_1) \otimes \mathrm{H}^*(\mathcal{G}\mathrm{r}, \mathcal{A}_2)$.

(6.2b) $R(r_2)_*(\tau^0(\mathcal{A}_1) *_X \tau^0(\mathcal{A}_2))|\Delta = \tau^0(\mathrm{H}^*(\mathcal{G}\mathrm{r}, \mathcal{A}_1 * \mathcal{A}_2))$

(6.2c) the sheaves $R^k(r_2)_*(\tau^0(\mathcal{A}_1) *_X \tau^0(\mathcal{A}_2))$ are constant .

The first two parts follow from (5.8). To prove (6.2c) we have to show that

$$R^k(r_2 \circ m)_* \tilde{\mathcal{B}} \text{ is constant;} \tag{6.3}$$

here $q^*\tilde{\mathcal{B}} = p^*(\tau^0(\mathcal{A}_1) \boxtimes \tau^0(\mathcal{A}_2))$. To this end we note that natural action of the group $\mathbb{G}_a \times \mathbb{G}_a$ by translations on $X \times X = \mathbb{A}^1 \times \mathbb{A}^1$ induces an action of $\mathbb{G}_a \times \mathbb{G}_a$ on $\mathcal{G}r_X \tilde{\times} \mathcal{G}r_X$. Clearly, the sheaf $\tilde{\mathcal{B}}$ is equivariant under this action. This implies (6.2c)

□

6.2. **Remark.** *The statements in* (6.2) *hold for an arbitrary curve* X. *This can be seen by utilizing the* $\operatorname{Aut}(\mathcal{O})$*-torsor* $\hat{X} \to X$ *of remark 5.1 and proposition 2.2; for details see* [Na].

Let $\operatorname{Mod}_{\mathbb{k}}$ denote the category of finite dimensional vector spaces over $\mathbb{k}$. To make $\mathrm{H}^* : \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}r, \mathbb{k}) \to \operatorname{Mod}_{\mathbb{k}}$ into a tensor functor we alter, following Beilinson and Drinfeld, the commutativity constraint of the previous section slightly. We consider the constraint from §3 on the category $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}r, \mathbb{k}) \otimes \operatorname{Mod}_{\mathbb{k}}^{\epsilon}$ and restrict it to a subcategory that we identify with $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}r, \mathbb{k})$. Divide $\mathcal{G}r$ into unions of connected components $\mathcal{G}r = \mathcal{G}r_+ \cup \mathcal{G}r_-$ so that the dimension of $G_{\mathcal{O}}$-orbits is even in $\mathcal{G}r_+$ and odd in $\mathcal{G}r_-$. This gives a $\mathbb{Z}_2$-grading on the category $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}r, \mathbb{k})$ hence a new $\mathbb{Z}_2$-grading on $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}r, \mathbb{k}) \otimes \operatorname{Mod}_{\mathbb{k}}^{\epsilon}$. The subcategory of even objects is identified with $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}r, \mathbb{k})$ by forgetting the grading. Hence, we conclude from the previous lemma:

6.3. **Proposition.** *The functor* $\mathrm{H}^* : \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}r, \mathbb{k}) \to \operatorname{Mod}_{\mathbb{k}}$ *is a tensor functor with respect to the above commutativity constraint.*

Let us write $\operatorname{Mod}_{\mathbb{k}}(X_*(T)))$ for the (tensor) category of finitely generated $\mathbb{k}$-modules with a $X_*(T)$-grading. We can view $F = \oplus_{\nu \in X_*(T)} F_\nu$ as a functor from $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}r, \mathbb{k})$ to $\operatorname{Mod}_{\mathbb{k}}(X_*(T)))$. Then we have the following generalization of the previous proposition:

6.4. **Proposition.** *The functor* $F : \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}r, \mathbb{k}) \to \operatorname{Mod}_{\mathbb{k}}(X_*(T)))$ *is a tensor functor.*

Proof. The notion of the subspaces S_ν and T_ν can be extended to the situation of families, i.e., to the global Grassmannians $\mathcal{G}r_{X^n}$. Recall that the fiber of the projection $r_n : \mathcal{G}r_{X^n} \to X^n$ over the point $(x_1, \dots, x_n)$ is simply $\prod_{i=1}^k \mathcal{G}r_{y_i}$, where $\{y_1, \dots, y_k\} = \{x_1, \dots, x_n\}$, with all the y_i distinct. Attached to the coweight $\nu \in X_*(T)$ we associate the ind-subscheme

$$\prod_{\nu_1 + \dots + \nu + k = \nu} S_{\nu_i} \subset \prod_{i=1}^k \mathcal{G}r_{y_i} = r_n^{-1}(x_1, \dots, x_n) \tag{6.4}$$

These ind-schemes altogether form an ind-subscheme $S_\nu(X^n)$ of $\mathcal{G}r_{X^n}$. This is easy to see for $n = 1$ by choosing a global parameter, for example. By the same argument we see that outside of the diagonals $S_\nu(X^n)$ form a subscheme. It is now not difficult to check that the closure of this locus lies inside $S_\nu(X^n)$. Similarly, we define the ind-subschemes $T_\nu(X^n)$. Let us write s_ν and t_ν for the inclusion maps of $S_\nu(X^n)$ and $T_\nu(X^n)$ to $\mathcal{G}r_{X^n}$, respectively. We have the action of $\mathbb{G}_m$ on $\mathcal{G}r_{X^n}$ via the cocharacter $2\check{\rho}$. The fixed point set of this action consists of the locus of products of the fixed points in the individual affine Grassmannians, i.e., above the point $(x_1,\dots,x_n)$ where $\{x_1,\dots,x_n\}=\{y_1,\dots,y_k\}$, with all the y_i distinct, the fixed points are of of the form

$$(L_{\nu_1},\dots,L_{\nu_k})\in\prod_{i=1}^{k}\mathcal{G}\mathrm{r}_{y_i}; \tag{6.5}$$

recall that we write L_ν for the point in $\mathcal{G}\mathrm{r}$ corresponding to the cocharacter $\nu\in X_*(T)$. We write C_ν for the subset of the fixed point locus lying inside $S_\nu(X^n)$, i.e.,

$$C_\nu\cap r_n^{-1}(x_1,\dots,x_n)\ =\ \bigcup_{\nu_1+\cdots+\nu+k=\nu}\{(L_{\nu_1},\dots,L_{\nu_k})\}\,. \tag{6.6}$$

Let us write $i_\nu: S_\nu(X^n)\to\mathcal{G}\mathrm{r}_{X^n}$ and $k_\nu: T_\nu(X^n)\to\mathcal{G}\mathrm{r}_{X^n}$ for the inclusions. By the same argument as in the proof of theorem 3.2 we see that

$$S_\nu(X^n)\ =\ \{z\in\mathcal{G}\mathrm{r}_{X^n}\mid\lim_{s\to0}2\check\rho(s)z\in C_\nu\} \tag{6.7}$$

and

$$T_\nu(X^n)\ =\ \{z\in\mathcal{G}\mathrm{r}_{X^n}\mid\lim_{s\to\infty}2\check\rho(s)z\in C_\nu\}\,. \tag{6.8}$$

Let us write $p_\nu: S_\nu(X^n)\to C_\nu$ and $q_\nu: T_\nu(X^n)\to C_\nu$ for the retractions:

$$p_\nu(z)\ =\ \lim_{s\to0}2\check\rho(s)z\qquad\text{for }\ z\in S_\nu(X^n) \tag{6.9}$$

$$q_\nu(z)\ =\ \lim_{s\to\infty}2\check\rho(s)z\qquad\text{for }\ z\in T_\nu(X^n)\,. \tag{6.10}$$

Furthermore,

$$C_\nu\ =\ S_\nu(X^n)\cap T_\nu(X^n)\,. \tag{6.11}$$

By Theorem 1 of [Br] we conclude that

$$i_\nu^!s_\nu^*\mathcal{B}\ =\ k_\nu^*t_\nu^!\mathcal{B}\qquad\text{for }\ \mathcal{B}\in\mathrm{P}_{G_{X^n,\mathcal{O}}}(\mathcal{G}\mathrm{r}_{X^n},\Bbbk)\,. \tag{6.12}$$

Rephrasing this result in terms of the contractions p_ν and q_ν:

$$R(p_\nu)_!s_\nu^*\mathcal{B}\ =\ R(q_\nu)_*t_\nu^!\mathcal{B}\qquad\text{for }\ \mathcal{B}\in\mathrm{P}_{G_{X^n,\mathcal{O}}}(\mathcal{G}\mathrm{r}_{X^n},\Bbbk)\,. \tag{6.13}$$

Let us now, for simplicity, choose $X=\mathbb{A}^1$. Let $\mathcal{A}_1,\mathcal{A}_2\in\mathrm{P}_{G_\mathcal{O}}(\mathcal{G}\mathrm{r},\Bbbk)$. We write $\mathcal{B}_1=\tau^0\mathcal{A}_1$ and $\mathcal{B}_2=\tau^0\mathcal{A}_2$. We form the convolution product $\mathcal{B}_1*\mathcal{B}_2=Rm_*\tilde{\mathcal{B}}$. By statement (6.2c) we see:

$$\begin{gathered}\text{The sheaf }\ R^k(r_2)_*Rm_*\tilde{\mathcal{B}}\ \text{ on }X^2\text{ is constant with fiber}\\ \mathrm{H}^k(\mathcal{G}\mathrm{r},\mathcal{A}_1*\mathcal{A}_2)\ =\ \bigoplus_{k_1+k_2=k}\mathrm{H}^{k_1}(\mathcal{G}\mathrm{r},\mathcal{A}_1)\otimes\mathrm{H}^{k_2}(\mathcal{G}\mathrm{r},\mathcal{A}_2)\,.\end{gathered} \tag{6.14}$$

Let us now consider the sheaves

$$\mathcal{L}^k(\mathcal{A}_1,\mathcal{A}_2)=R^k(r_2)_*R(p_\nu)_!s_\nu^*Rm_*\tilde{\mathcal{B}}=R^k(r_2)_*R(q_\nu)_*t_\nu^!Rm_*\tilde{\mathcal{B}}\,; \tag{6.15}$$

here we have used (6.13) to identify the last two sheaves. It is now easy to calculate the fibers of the sheaf $\mathcal{L}^k(\mathcal{A}_1,\mathcal{A}_2)$ using theorem 3.6. First of all,

$$\mathcal{L}^k(\mathcal{A}_1,\mathcal{A}_2)_{(x_1,x_2)}\ =\ 0\qquad\text{if }k\neq2\rho(\nu)\,. \tag{6.16}$$

Secondly,

$$\mathcal{L}^k(\mathcal{A}_1,\mathcal{A}_2)_{(x_1,x_2)} = \begin{cases} \mathrm{H}_c^{2\rho(\nu)}(S_\nu, \mathcal{A}_1 * \mathcal{A}_2) & \text{if } x_1 = x_2 \\ \oplus_{\nu_1+\nu_2=\nu}\mathrm{H}_c^{2\rho(\nu_1)}(S_{\nu_1},\mathcal{A}_1)\otimes \mathrm{H}_c^{2\rho(\nu_2)}(S_{\nu_2},\mathcal{A}_2) & \text{if } x_1 \neq x_2\,. \end{cases} \tag{6.17}$$

We now proceed as in the proof of theorem 3.6. Let us consider the closures $\bar{S}_\nu(X^n)$ and $\bar{T}_\nu(X^n)$ of the ind-subschemes $S_\nu(X^n)$ and $T_\nu(X^n)$ and let us write $\bar{i}_\nu : \bar{S}_\nu(X^n) \to \mathcal{G}\mathrm{r}_{X^n}$ and $\bar{k}_\nu : \bar{T}_\nu(X^n) \to \mathcal{G}\mathrm{r}_{X^n}$ for the inclusions. Let us write $\mathcal{B} = Rm_*\tilde{\mathcal{B}}$. Then we have the following canonical morphisms

$$R(r_2)_*\mathcal{B} = R(r_2)_!\mathcal{B} \to R(r_2)_!\bar{s}_\nu^*\mathcal{B} \tag{6.18a}$$

$$R(r_2)_*\mathcal{B} \leftarrow R(r_2)_* t_\nu^!\mathcal{B}\,. \tag{6.18b}$$

These morphisms give us two filtrations of $R^k(r_2)_*Rm_*\tilde{\mathcal{B}}$, one by kernels of the the morphisms

$$R^k(r_2)_*\mathcal{B} \to R^k(r_2)_!\bar{s}_\nu^*\mathcal{B} \tag{6.19}$$

and the other by images of the morphisms

$$R^k(r_2)_*\bar{t}_\nu^!\mathcal{B} \to R^k(r_2)_*\mathcal{B}\,. \tag{6.20}$$

By the discussion above, these filtrations are complementary and hence yield the following canonical isomorphism

$$R^k(r_2)_*Rm_*\tilde{\mathcal{B}} \;=\; \bigoplus_{2\rho(\nu)=k} \mathcal{L}^k(\mathcal{A}_1,\mathcal{A}_2)\,. \tag{6.21}$$

By (6.14) the sheaf on the right hand side is constant. Therefore the sheaves $\mathcal{L}^k(\mathcal{A}_1,\mathcal{A}_2)$ must be constant. Appealing to (6.17) completes the proof.

□

7. The case of a field of characteristic zero

In this section we treat the case when the base ring $\Bbbk$ is a field of characteristic zero. This case was treated already in [Gi] when $\Bbbk = \mathbb{C}$. Here we make use of Tannakian formalism, using [DM] as a general reference. In section 11, where we work over an arbitrary base ring $\Bbbk$, we carry out the constructions explicitly without referring to the general Tannakian formalism.

7.1. **Lemma.** *If $\Bbbk$ is a field of characteristic zero then the category $\mathrm{P}_{G_\mathcal{O}}(\mathcal{G}r, \Bbbk)$ is semisimple. In particular, the sheaves $\mathcal{I}_!(\lambda,\Bbbk)$, $\mathcal{I}_{!*}(\lambda,\Bbbk)$, and $\mathcal{I}_*(\lambda,\Bbbk)$ are isomorphic.*

Proof. The parity vanishing of the stalks of $\mathcal{I}_{!*}(\lambda,\Bbbk)$, proved in [Lu], section 11, and the fact that the orbits $\mathcal{G}\mathrm{r}^\lambda$ are simply connected implies immediately that there are no extensions between the simple objects in $\mathrm{P}_{G_\mathcal{O}}(\mathcal{G}\mathrm{r}, \Bbbk)$. □

7.2. **Remark.** *The use of the above lemma can be avoided. One must then ignore this section and first go through the rest of the paper in the case when $\Bbbk$ is a field of characteristic zero. The arguments of section 12, in a greatly simplified form, then give theorem 7.3.*

By [DM], proposition 1.20, $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r},\Bbbk)$ is a neutral Tannakian category with fiber functor F. Hence, by theorem 2.11 of [DM], we conclude:

$$\begin{gathered}\text{there is a group scheme } \tilde{G} \text{ over } \Bbbk \text{ such that}\\ \text{the category of finite dimensional } \Bbbk\text{-representations of } \tilde{G}\\ \text{is equivalent to } \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r},\Bbbk), \text{ as tensor categories}\,. \end{gathered} \tag{7.1}$$

We will now identify the group $\tilde{G}$. Let us write $\check{G}$ for the dual group of G, i.e., $\check{G}$ is the split reductive group over $\Bbbk$ whose root datum is dual to that of G.

7.3. **Theorem.** *The category of finite dimensional $\Bbbk$-representations of $\check{G}$ is equivalent to $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}r,\Bbbk)$, as tensor categories.*

Before giving a proof of this theorem we discuss it briefly from the point of view of representation theory. We can view the theorem as giving us a geometric interpretation of representation theory of $\check{G}$. First of all, as we use global cohomology as fiber functor, it follows that the representation space for the representation $V_{\mathcal{F}}$, associated to $\mathcal{F} \in \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r},\Bbbk)$, is the global cohomology $\mathrm{H}^*(\mathcal{G}\mathrm{r},\mathcal{F})$. As the proof of the theorem will show, $\check{T}$, the dual torus of T, is a maximal torus in $\check{G}$. The decomposition of $V_{\mathcal{F}}$ into its $\check{T}$-weight spaces is then given by theorem 3.6:

$$\mathrm{H}^*(\mathcal{G}\mathrm{r},\mathcal{F}) \;\cong \bigoplus_{\nu\in X^*(\check{T})} \mathrm{H}_c^{2\rho(\nu)}(S_\nu,\mathcal{F})\,. \tag{7.2}$$

Given a dominant $\lambda \in X_*(T) = X^*(\check{T})$ we can associate to it both the highest weight representation $L(\lambda)$ of $\check{G}$ and the sheaf $\mathcal{I}_{!*}(\lambda,\Bbbk) \in \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r},\Bbbk)$. Obviously, $V_{\mathcal{I}_{!*}(\lambda,\Bbbk)}$ is irreducible and by the formula above we see that it is of highest weight λ. Hence, $V_{\mathcal{I}_{!*}(\lambda,\Bbbk)} = L(\lambda)$. Combining this discussion with lemma 7.1 and proposition 3.10 gives:

7.4. **Corollary.** *The ν-weight space $L(\lambda)_\nu$ of $L(\lambda)$ can be canonically identified with the $\Bbbk$-vector space spanned by the irreducible components of $\overline{\mathcal{G}r_\lambda} \cap S_\nu$. In particular, the dimension of $L(\lambda)_\nu$ is given by the number of irreducible components of $\overline{\mathcal{G}r_\lambda} \cap S_\nu$.*

The rest of this section is devoted to the proof of theorem 7.3. We begin with an observation:

$$\text{the group scheme } \tilde{G} \text{ is a split connected reductive algebraic group} \tag{7.3}$$

To see that $\tilde{G}$ is algebraic, we observe that it has a tensor generator. Let $\lambda_1,\dots,\lambda_r$ be a set of generators for the dominant weights in $X_*(T)$. As a generator we can then take $\oplus\mathcal{I}_{!*}(\lambda_i,\Bbbk)$. It is tensor generator because for any dominant λ the sheaf $\mathcal{I}_{!*}(\lambda,\Bbbk)$ appears as a direct summand in the product

$$\mathcal{I}_{!*}(\lambda_1,\Bbbk)^{*k_1} * \cdots * \mathcal{I}_{!*}(\lambda_r,\Bbbk)^{*k_r}\,; \tag{7.4}$$

here $\lambda = \sum k_1\lambda_1 + \cdots + k_r\lambda_r$. Thus, by [DM], proposition 2.20, $\tilde{G}$ is an algebraic group. As there is no tensor subcategory of $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r},\Bbbk)$ whose objects are direct sums of finitely many fixed irreducible objects the group $\tilde{G}$ is connected by [DM], corollary 2.22. Finally, as $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r},\Bbbk)$ is semisimple, $\tilde{G}$ is reductive, by [DM], proposition 2.23.

To see that $\tilde{G}$ is split, we exhibit a split maximal torus in $\tilde{G}$. By proposition 6.4 the fiber functor $F = \mathrm{H}^*$ factors as follows:

$$F = \mathrm{H}^* : \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}r, \Bbbk) \to \mathrm{Mod}_{\Bbbk}(X_*(T))) \to \mathrm{Mod}_{\Bbbk}\,. \tag{7.5}$$

This gives us a homomorphism $\check{T} \to \tilde{G}$; here $\check{T}$ is the torus dual to T. As any character $\lambda \in X^*(\check{T}) = X_*(T)$ appears as the direct summand $F_\lambda(\mathcal{J}_{!*}(\lambda, \Bbbk))$ in $F(\mathcal{J}_{!*}(\lambda, \Bbbk))$ we conclude that $\check{T}$ is a split torus in $\tilde{G}$. It is clearly maximal as the representation ring of $\tilde{G}$ is of the same rank as $\check{T}$.

It now remains to identify the root datum of $\tilde{G}$ with the dual of the root datum of G. Recall that we have also fixed a choice of positive roots, i.e., a Borel B such that $T \subset B \subset G$. The root datum of G is then given as $(X^*(T), X_*(T), \Phi(G,T), \check{\Phi}(G,T))$, where $\Phi(G,T) \subset X^*(T)$ are the roots and $\check{\Phi}(G,T) \subset X_*(T)$ are the coroots of G with respect to T. Because $X^*(\check{T}) = X_*(T)$ and $X_*(\check{T}) = X^*(T)$, it suffices to show that

$$\Phi(\tilde{G}, \check{T}) = \check{\Phi}(G,T) \quad \text{and} \quad \check{\Phi}(\tilde{G}, \check{T}) = \Phi(G,T)\,. \tag{7.6}$$

To this end we note that theorem 3.2, corollary 3.3, and proposition 3.10 imply that:

(7.7) The irreducible representations of $\tilde{G}$ are parametrized by dominant coweights $\lambda \in X_*(T)$.

and

(7.8) The $\check{T}$-weights of the irreducible representation associated to λ are the same as the $\check{T}$-weights of the irreducible representation of $\check{G}$ associated to λ.

Note that the choice of a Borel subgroup of $\tilde{G}$ is equivalent to a consistent choice of a line, the highest weight line, in each irreducible representation of $\tilde{G}$. The choice $F_\lambda(\mathcal{J}_{!*}(\lambda, \Bbbk))$ in $F(\mathcal{J}_{!*}(\lambda, \Bbbk))$ for all dominant $\lambda \in X_*(T)$ yields a Borel subgroup $\tilde{B}$ of $\tilde{G}$ such that the dominant weights of $\tilde{G}$ in $X^*(\check{T})$ coincide with the dominant coweights of G in $X_*(T)$. This implies that the simple coroot directions of the triple $(\check{T}, \tilde{B}, \tilde{G})$ coincide with the simple root directions of (T, B, G). The statements (7.7), (7.8) above now imply that the simple roots of the triple $(\check{T}, \tilde{B}, \tilde{G})$ coincide with the simple coroots of (T, B, G). This, finally gives (7.6).

8. **Standard sheaves**

In this section we prove some basic results about standard sheaves which will be crucial for us later. Let us write $\mathbb{D}$ for the Verdier duality functor.

8.1. **Proposition.** *We have*

(a) $$\mathcal{J}_!(\lambda, \Bbbk) \;\cong\; \mathcal{J}_!(\lambda, \mathbb{Z}) \overset{L}{\underset{\mathbb{Z}}{\otimes}} \Bbbk$$

(b) $$\mathcal{J}_*(\lambda, \Bbbk) \;\cong\; \mathcal{J}_*(\lambda, \mathbb{Z}) \overset{L}{\underset{\mathbb{Z}}{\otimes}} \Bbbk$$

(c) $$\mathbb{D}\,\mathcal{J}_!(\lambda, \Bbbk) \;\cong \mathcal{J}_*(\lambda, \Bbbk)\,.$$

Proof. The proofs of (a) and (b) are analogous and hence we will only prove (a). Because

$$\mathrm{H}_c^*(S_\nu, \mathcal{J}_!(\lambda,\mathbb{Z})\overset{L}{\underset{\mathbb{Z}}{\otimes}}\Bbbk) \;=\; \mathrm{H}_c^*(S_\nu, \mathcal{J}_!(\lambda,\mathbb{Z}))\overset{L}{\underset{\mathbb{Z}}{\otimes}}\Bbbk \tag{8.1}$$

and, by proposition 3.10, $\mathrm{H}_c^*(S_\nu, \mathcal{J}_!(\lambda,\mathbb{Z}))$ is a free abelian group in degree $2\rho(\nu)$ we conclude that $\mathrm{H}_c^*(S_\nu, \mathcal{J}_!(\lambda,\mathbb{Z})\overset{L}{\underset{\mathbb{Z}}{\otimes}}\Bbbk)$ is nonzero only in degree $2\rho(\nu)$. Hence, by lemma 3.9, we see that $\mathcal{J}_!(\lambda,\mathbb{Z})\overset{L}{\underset{\mathbb{Z}}{\otimes}}\Bbbk$ is perverse. There is a canonical map

$$\mathcal{J}_!(\lambda,\Bbbk) \xrightarrow{\iota} \mathcal{J}_!(\lambda,\mathbb{Z})\overset{L}{\underset{\mathbb{Z}}{\otimes}}\Bbbk\,, \tag{8.2}$$

which is an isomorphism when restricted to $\mathcal{G}\mathrm{r}^\lambda$. Therefore, applying the functor F_ν to the morphism ι and using the previous proposition yields an isomorphism

$$F_\nu(\mathcal{J}_!(\lambda,\Bbbk)) = \Bbbk[Irr(\overline{\mathcal{G}\mathrm{r}}_\lambda\cap S_\nu)] \xrightarrow{F_\nu(\iota)} \mathbb{Z}[Irr(\overline{\mathcal{G}\mathrm{r}}_\lambda\cap S_\nu)]\overset{L}{\underset{\mathbb{Z}}{\otimes}}\Bbbk \;=\; F_\nu(\mathcal{J}_!(\lambda,\mathbb{Z})\overset{L}{\underset{\mathbb{Z}}{\otimes}}\Bbbk)\,. \tag{8.3}$$

By corollary 3.7 the functor $F = \oplus F_\nu$ is faithful and thus we conclude that ι is an isomorphism.

The proof of part (c) proceeds in a similar fashion. First we observe that

$$\mathrm{H}_{T_\nu}^*(\mathcal{G}\mathrm{r}, \mathbb{D}\,\mathcal{J}_!(\lambda,\Bbbk)) \cong \mathbb{D}(\mathrm{H}_c^*(T_\nu, \mathcal{J}_!(\lambda,\Bbbk))) = \mathbb{D}(\mathrm{H}_c^*(S_{w_0\cdot\nu}, \mathcal{J}_!(\lambda,\Bbbk)))\,. \tag{8.4}$$

Because $\mathrm{H}_c^*(S_{w_0\cdot\nu}, \mathcal{J}_!(\lambda,\Bbbk))$ is a free $\Bbbk$-module concentrated in degree $2\rho(w_0\cdot\nu)$, we conclude that $\mathbb{D}(\mathrm{H}_{T_{w_0\cdot\nu}}^*(\mathcal{G}\mathrm{r}, \mathcal{J}_!(\lambda,\Bbbk)))$ is a concentrated in degree $-2\rho(w_0\cdot\nu) = 2\rho(\nu)$. Thus, we conclude that $\mathbb{D}\,\mathcal{J}_!(\lambda,\Bbbk)$ is perverse. Furthermore, we note that

$$\begin{array}{ccc} \mathrm{H}_{T_\nu}^*(\mathcal{G}\mathrm{r}, \mathbb{D}\,\mathcal{J}_!(\lambda,\Bbbk)) & \xrightarrow{\cong} & \mathbb{D}(\mathrm{H}_c^*(T_\nu, \mathcal{J}_!(\lambda,\Bbbk))) \\ \big\downarrow & & \big\downarrow{\scriptstyle\cong} \\ \mathrm{H}_{T_\nu}^*(\mathcal{G}\mathrm{r}^\lambda, \mathbb{D}\,\mathcal{J}_!(\lambda,\Bbbk)) & \xrightarrow{\cong} & \mathbb{D}(\mathrm{H}_c^*(T_\nu\cap\mathcal{G}\mathrm{r}^\lambda, \mathcal{J}_!(\lambda,\Bbbk)))\,, \end{array} \tag{8.5}$$

which implies that the left hand arrow is also an isomorphism. We have a canonical map

$$\mathbb{D}\,\mathcal{J}_!(\lambda,\Bbbk) \xrightarrow{\iota} \mathcal{J}_*(\lambda,\Bbbk) \tag{8.6}$$

To show that this map is an isomorphism it suffice to show that the maps $F_\nu(\iota)$ are isomorphisms. Restricting to $\mathcal{G}\mathrm{r}^\lambda$ gives us the following commutative diagram:

$$\begin{array}{ccc} \mathrm{H}_{T_\nu}^*(\mathcal{G}\mathrm{r}, \mathbb{D}\,\mathcal{J}_!(\lambda,\Bbbk)) & \xrightarrow{F_\nu(\iota)} & \mathrm{H}_{T_\nu}^*(\mathcal{G}\mathrm{r}, \mathcal{J}_*(\lambda,\Bbbk)) \\ {\scriptstyle\cong}\big\downarrow & & \big\downarrow{\scriptstyle\cong} \\ \mathrm{H}_{T_\nu}^*(\mathcal{G}\mathrm{r}^\lambda, \mathbb{D}\,\mathcal{J}_!(\lambda,\Bbbk)) & \xrightarrow{\cong} & \mathrm{H}_{T_\nu}^*(\mathcal{G}\mathrm{r}^\lambda, \mathcal{J}_*(\lambda,\Bbbk))\,. \end{array} \tag{8.7}$$

In this diagram the bottom arrow is an isomorphism because ι restricted to $\mathcal{G}\mathrm{r}^\lambda$ is an isomorphism, the left vertical arrow is an isomorphism by (8.5), and finally, the right vertical arrow is an isomorphism by proposition 3.10 (or, rather, by the proof thereof). This shows that $F_\nu(\iota)$ is an isomorphism.

□

8.2. **Proposition.** *The canonical map* $\mathcal{J}_!(\lambda,\mathbb{Z}) \to \mathcal{J}_{!*}(\lambda,\mathbb{Z})$ *is an isomorphism.*

Proof. Let us consider the following commutative diagram:

$$\begin{array}{ccc} \mathcal{J}_!(\lambda,\mathbb{Z}) & \xrightarrow{\alpha} & \mathcal{J}_*(\lambda,\mathbb{Z}) \\ \downarrow & & \downarrow \\ \mathcal{J}_!(\lambda,\mathbb{Q}) & \longrightarrow & \mathcal{J}_*(\lambda,\mathbb{Q}) . \end{array} \tag{8.8}$$

The bottom map is an isomorphism by lemma 7.1. Let us apply the functor F_ν to this diagram. The columns become inclusions by corollary 8.1 and the bottom arrow is an isomorphism as we just observed. Therefore, $F_\nu(\alpha)$ is an inclusion and therefore so is α. This implies that the canonical surjection $\mathcal{J}_!(\lambda,\mathbb{Z}) \twoheadrightarrow \mathcal{J}_{!*}(\lambda,\mathbb{Z})$ is an isomorphism. □

9. Representability of the weight functors

In section §3 we showed that the functors $F_\nu \overset{def}{=} \mathrm{H}_c^{2\rho(\nu)}(S_\nu, \ ) \cong \mathrm{H}^*_{T_\nu}(\mathcal{G}r, -)[2\rho(\nu)]$, from $\mathrm{P}_{G_\mathcal{O}}(\mathcal{G}r, \Bbbk)$ to $\mathrm{Mod}_\Bbbk$, for $\nu \in X_*(T)$, are exact. Hence, one would expect them to be (pro) representable. Here we prove that this is indeed the case:

9.1. **Proposition.** *Let $Z \subset \mathcal{G}r$ be a closed subset which is a finite union of $G_\mathcal{O}$-orbits. The functor F_ν restricted to $\mathrm{P}_{G_\mathcal{O}}(Z, \Bbbk)$ is represented by a projective object $P_Z(\nu, \Bbbk)$ of $\mathrm{P}_{G_\mathcal{O}}(Z, \Bbbk)$.*

Proof. We make use of the induction functors. Let us recall their construction. For more details see, for example, [MiV1]. Let A be an algebraic group acting on a variety Y and let B be a subgroup of A. The forgetful functor $\mathcal{F}^A_B : \ D_A(Y,\Bbbk) \to D_B(Y,\Bbbk)$ has a left adjoint $\gamma^A_B : \ D_B(Y,\Bbbk) \to D_A(Y,\Bbbk)$ which can be constructed as follows. Consider the diagram

$$Z \xleftarrow{\ p\ } A \times Z \xrightarrow{\ q\ } A \times_B Z \xrightarrow{\ a\ } Z . \tag{9.1}$$

The maps p and q are projections, and a is the action map. The group $A \times B$ acts on the leftmost copy of Z via the factor B, on $A \times Z$ and $A \times_B Z$ by the formula $(a,b){\cdot}(a',z) = (a{\cdot}a'{\cdot}b^{-1}, b{\cdot}z)$, and on the leftmost copy of Z via the factor A. The left adjoint γ^A_B is now given by

$$\gamma^A_B(\mathcal{A}) \ = \ a_!\tilde{\mathcal{A}} \ \text{ where } \ \tilde{\mathcal{A}} \ \text{ is defined via } \ q^!\tilde{\mathcal{A}} \ \cong \ p^!\mathcal{A}\,; \text{for } \ \mathcal{A} \in D_B(Y,\Bbbk) \tag{9.2}$$

Let $\mathcal{O}_n = \mathcal{O}/z^{n+1}$ and let us write $G_{\mathcal{O}_n}$ for the algebraic group whose $\mathbb{C}$-points are $G(\mathcal{O}_n)$. We use analogous notation for other groups. Now choose $n >> 0$ so that the $G_\mathcal{O}$-action on Z factors through the action of $G_{\mathcal{O}_n}$. We write $P_Z(\nu,\Bbbk) = {}^p\mathrm{H}^0(\gamma^{G_{\mathcal{O}_n}}_{\{e\}}(\Bbbk_{T_\nu \cap Z}[-2\rho(\nu)]))$. We claim:

$$\text{the functor } F_\nu : \mathrm{P}_{G_\mathcal{O}}(Z,\Bbbk) \to \mathrm{Mod}_\Bbbk \text{ is represented by } P_Z(\nu,\Bbbk) \tag{9.3}$$

To see this, let $\mathcal{A} \in \mathrm{P}_{G_\mathcal{O}}(Z,\Bbbk)$, and then:

$$(9.4)\quad F_\nu(\mathcal{A}) \;=\; \mathrm{H}^{2\rho(\nu)}_{T_\nu}(Z,\mathcal{A}) \;=\; \mathrm{Ext}^0_{\mathrm{D}(Z,\Bbbk)}(\Bbbk_{T_\nu\cap Z}[-2\rho(\nu)], \mathcal{F}^{G_{\mathcal{O}_n}}_{\{e\}}\ \mathcal{A}) \;\cong\; \mathrm{Ext}^0_{\mathrm{D}_{G_{\mathcal{O}_n}}(Z,\Bbbk)}(\gamma^{G_{\mathcal{O}_n}}_{\{e\}}\Bbbk_{T_\nu\cap Z}[-2\rho(\nu)],\ \mathcal{A})\,.$$

Let us write $\mathcal{F} = \gamma^{G_{\mathcal{O}_n}}_{\{e\}}\Bbbk_{T_\nu\cap Z}[-2\rho(\nu)]$. Then

$$(9.5)\qquad \text{the sheaf } \mathcal{F} \text{ lies in } {}^p\mathrm{D}^{\leq 0}_{G_{\mathcal{O}_n}}(Z,\Bbbk)\,.$$

To see this, let us write d for the largest integer such that ${}^p\mathrm{H}^d(\mathcal{F}) \neq 0$. Then, we see, as in (9.4), that

$$(9.6)\quad 0 \neq \mathrm{Hom}_{\mathrm{D}_{G_{\mathcal{O}_n}}(Z,\Bbbk)}(\mathcal{F}, {}^p\mathrm{H}^d(\mathcal{F})[-d]) = \mathrm{Ext}^{-d}_{\mathrm{D}_{G_{\mathcal{O}_n}}(Z,\Bbbk)}(\mathcal{F}, {}^p\mathrm{H}^d(\mathcal{F})) \cong \mathrm{H}^{2\rho(\nu)-d}_{T_\nu}(Z,\mathcal{A})\,.$$

This forces $d = 0$ and we have proved (9.5). This immediately implies (9.3).

□

9.2. **Corollary.** *The category* $\mathrm{P}_{G_\mathcal{O}}(Z,\Bbbk)$ *has enough projectives.*

Proof. Let $\mathcal{A} \in \mathrm{P}_{G_\mathcal{O}}(Z,\Bbbk)$. Choose finitely generated $\Bbbk$-projective covers $f_\nu : P_\nu \to F_\nu(\mathcal{A})$. Then

$$(9.7)\quad \mathrm{Hom}(P_\nu \otimes_\Bbbk P_Z(\nu,\Bbbk), \mathcal{A}) \;\cong\; \mathrm{Hom}_\Bbbk[P_\nu, \mathrm{Hom}(P_Z(\nu,\Bbbk),\mathcal{A})] \;\cong\; \mathrm{Hom}_\Bbbk[P_\nu, F_\nu(\mathcal{A})]\,.$$

By construction, the map $p_\nu \in \mathrm{Hom}(P_\nu \otimes_\Bbbk P_Z(\nu,\Bbbk),\mathcal{A})$ corresponding to the $\Bbbk$-projective cover $f_\nu : P_\nu \to F_\nu(\mathcal{A})$ satisfies $F_\nu(p_\nu) = f_\nu$. Since $\oplus_\nu F_\nu = F$ is exact and faithful, $\oplus_\nu\ P_\nu \otimes_\Bbbk P_Z(\nu,\Bbbk)$ is a projective cover of $\mathcal{A}$.

□

We can describe the sheaf $P_Z(\nu,\Bbbk) = {}^p\mathrm{H}^0(\gamma^{G_{\mathcal{O}_n}}_{\{e\}}(\Bbbk_{T_\nu\cap Z}[-2\rho(\nu)]))$ rather explicitly as follows. Let us consider the following diagram:

$$(9.8)\qquad \begin{array}{ccccc} Z & \xleftarrow{\;p\;} & G_{\mathcal{O}_n} \times Z & \xrightarrow{\;a\;} & Z \\ {\scriptstyle i}\uparrow & & \uparrow & & \| \\ T_\nu \cap Z & \xleftarrow{\;r\;} & G_{\mathcal{O}_n} \times (T_\nu \cap Z) & \xrightarrow{\;\tilde a\;} & Z \end{array}$$

and use it to calculate $\gamma^{G_{\mathcal{O}_n}}_{\{e\}}(\Bbbk_{T_\nu\cap Z}[-2\rho(\nu)])$:

$$(9.9)\quad \gamma^{G_{\mathcal{O}_n}}_{\{e\}}(\Bbbk_{T_\nu\cap Z}[-2\rho(\nu)]) = Ra_!p^!i_*\Bbbk_{T_\nu\cap Z}[-2\rho(\nu)] = R\tilde a_!r^!\Bbbk_{T_\nu\cap Z}[-2\rho(\nu)] = R\tilde a_!\Bbbk_{G_{\mathcal{O}_n}\cap(T_\nu\cap Z)}[2\dim(G_{\mathcal{O}_n}) - 2\rho(\nu)]\,.$$

Let us consider a point $L_\eta \in Z$, where we again choose $\eta \in X_*(T)$ dominant. Then

9.3. **Lemma.** *The dimension of the fiber* $\tilde a^{-1}(L_\eta)$ *is* $\dim G_{\mathcal{O}_n} - \rho(\nu+\eta)$ *if* $\eta \geq \nu$, *otherwise it is empty.*

Proof. We have

$$(9.10)\quad \tilde{a}^{-1}(L_\eta) \;=\; \{(g,z)\in G_{\mathcal{O}_n}\times (T_\nu\cap Z)\mid g\cdot z=\eta\} \;=\; \{(g,z)\in G_{\mathcal{O}_n}\times (T_\nu\cap \mathcal{G}\mathrm{r}^\eta)\mid g\cdot z=\eta\}\,.$$

By theorem 3.2, we see that $\tilde{a}^{-1}(L_\eta)$ is empty unless $\eta\geq\nu$. Furthermore, from theorem 3.2, we see that if $\eta\geq\nu$ then

$$(9.11)\quad \dim \tilde{a}^{-1}(L_\eta) \;=\; \rho(\eta-\nu)+\dim((G_{\mathcal{O}_n})_\eta) \;=\; \rho(\eta-\nu)+\dim(G_{\mathcal{O}_n})-\dim\mathcal{G}\mathrm{r}^\eta \;=\; \rho(\eta-\nu)+\dim(G_{\mathcal{O}_n})-2\dim\rho(\eta) \;=\; \dim(G_{\mathcal{O}_n})-\rho(\eta+\nu)\,.$$

□

In other words, $P_Z(\nu,\Bbbk)$ is the zeroth perverse cohomology of the !-image of a (shift of) a constant sheaf under an "essentially semi-small" map. Here "essentially semi-small" means that the the dimensions of fibers have the correct increment but the generic fiber is not finite.

10. The structure of projectives that represent weight functors

In this section we analyze the projective $P_Z(\Bbbk)=\oplus_\nu\, P_Z(\nu,\Bbbk)$ which represents the fiber functor F on $\mathrm{P}_{G_\mathcal{O}}(Z,\Bbbk)$.

10.1. **Proposition.** *(a) Let $Y\subset Z$ be a closed subset consisting of $G_\mathcal{O}$-orbits. Then*

$$P_Y(\Bbbk) = {}^pH^0(P_Z(\Bbbk)\big|Y)\,,$$

and there is a canonical surjection

$$P_Z(\Bbbk)\to P_Y(\Bbbk)\,.$$

(b) The projective $P_Z(\Bbbk)$ has a filtration such that the the associated graded

$$Gr(P_Z(\Bbbk)) \;=\; \bigoplus_{\mathcal{G}r^\lambda\subseteq Z} F[\mathcal{J}_*(\lambda,\Bbbk)]^*\otimes_\Bbbk\mathcal{J}_!(\lambda,\Bbbk)\,.$$

In particular, $F(P_Z(\Bbbk))$ is free over $\Bbbk$.

(c) $P_Z(\nu,\Bbbk)\cong\; P_Z(\nu,\mathbb{Z})\overset{L}{\underset{\mathbb{Z}}{\otimes}}\Bbbk$.

Proof. We begin with (a). Let $i:\,Y\hookrightarrow Z$, the identity $\mathrm{Hom}(P_Z(\Bbbk),i_*-)=\mathrm{Hom}(i^*P_Z(\Bbbk),-)$ shows that the complex $P_Z(\Bbbk)\big|Y\in{}^pD^{\leq 0}_{G_\mathcal{O}}(Y,\Bbbk)$, represents F on the subcategory $\mathrm{P}_{G_\mathcal{O}}(Y,\Bbbk)$, and hence so does ${}^pH^0(P_Z(\Bbbk)\big|Y)\in\mathrm{P}_{G_\mathcal{O}}(Y,\Bbbk)$. Thus, $P_Y(\Bbbk)={}^pH^0(P_Z(\Bbbk)\big|Y)$. For any $\mathcal{A}\in\mathrm{P}_{G_\mathcal{O}}(Y,\Bbbk)$ we have the identity $\mathrm{Hom}(P_Z(\Bbbk),\mathcal{A})=\mathrm{Hom}(P_Y(\Bbbk),\mathcal{A})$. This gives a canonical map $P_Z(\Bbbk)\to P_Y(\Bbbk)$ which has to be a surjection.

Now we will prove parts (b) and (c) simultaneously. We will argue (b) by induction on the number of $G_\mathcal{O}$-orbits in Z. Let us assume that $\mathcal{G}\mathrm{r}^\lambda$ is an open $G_\mathcal{O}$-orbit in Z and let $Y=Z-\mathcal{G}\mathrm{r}^\lambda$. Let us consider the following exact sequence:

$$(10.1)\qquad 0\to K\to P_Z(\Bbbk)\to P_Y(\Bbbk)\to 0\,,$$

where K is simply the kernel of the canonical map from (a). Let M be a $\Bbbk$-module and let us take RHom of the exact sequence above to $\mathcal{I}_*(\lambda, M)$:

$$(10.2)\quad 0 \to \operatorname{Hom}(P_Y(\Bbbk), \mathcal{I}_*(\lambda, M)) \to \operatorname{Hom}(P_Z(\Bbbk), \mathcal{I}_*(\lambda, M)) \to \operatorname{Hom}(K, \mathcal{I}_*(\lambda, M)) \to \operatorname{Ext}^1(P_Y(\Bbbk), \mathcal{I}_*(\lambda, M)) .$$

By adjunction the first and last terms are zero and so we get, again by adjunction,

$$(10.3)\quad \operatorname{Hom}(K|_{\mathcal{G}r^\lambda}, M_{\mathcal{G}r^\lambda}[2\rho(\lambda)]) \cong \operatorname{Hom}(K, \mathcal{I}_*(\lambda, M)) \cong \operatorname{Hom}(P_Z(\Bbbk), \mathcal{I}_*(\lambda, M)) = F(\mathcal{I}_*(\lambda, M)) .$$

We can view (10.3) as a functor from $\Bbbk$-modules to $\Bbbk$-modules

$$(10.4)\qquad M \mapsto F(\mathcal{I}_*(\lambda, M)) .$$

This functor is, by the results in §8 represented by the free $\Bbbk$-module $F(\mathcal{I}_*(\lambda, \Bbbk))^*$. As it is also represented by $K|_{\mathcal{G}r^\lambda}$, we conclude:

$$(10.5)\qquad K|_{\mathcal{G}r^\lambda} \;=\; F(\mathcal{I}_*(\lambda, \Bbbk))^* \otimes_\Bbbk \Bbbk_{\mathcal{G}r^\lambda}[2\rho(\lambda)] .$$

Now we claim:

$$(10.6)\qquad K \;=\; F(\mathcal{I}_*(\lambda, \Bbbk))^* \otimes_\Bbbk \mathcal{I}_!(\lambda, \Bbbk) = \mathcal{I}_!(\lambda, F(\mathcal{I}_*(\lambda, \Bbbk))^*) .$$

To prove this claim, let us consider the following exact sequence

$$(10.7)\qquad 0 \to K' \to F(\mathcal{I}_*(\lambda, \Bbbk))^* \otimes_\Bbbk \mathcal{I}_!(\lambda, \Bbbk) \to K \to C \to 0 .$$

The kernel K' and the cokernel C are supported on Y. If we take RHom of the exact sequence (10.1) to C, we get

$$(10.8)\quad 0 \to \operatorname{Hom}(P_Y(\Bbbk), C) \xrightarrow{\alpha} \operatorname{Hom}(P_Z(\Bbbk), C) \to \operatorname{Hom}(K, C) \to \operatorname{Ext}^1(P_Y(\Bbbk), C) .$$

Because C is supported on Y the map α is an isomorphism and the last term vanishes. Therefore C must be zero. To prove that K' is zero, we first assume that $\Bbbk = \mathbb{Z}$. Then, by 8.2, $\mathcal{I}_!(\lambda, \mathbb{Z}) = \mathcal{I}_{!*}(\lambda, \mathbb{Z})$. As $\mathcal{I}_{!*}(\lambda, \mathbb{Z})$ has no subobjects supported on Y, $K' = 0$. Using (10.1) and (10.6), and proceeding by induction on the number of $G_\mathcal{O}$-orbits, we obtain (b) when $\Bbbk = \mathbb{Z}$.

We will now prove (c). Because $\Bbbk \overset{L}{\underset{\mathbb{Z}}{\otimes}} \mathcal{I}_!(\lambda, \mathbb{Z}) \cong \mathcal{I}_!(\lambda, \Bbbk)$ and because (b) holds for $\Bbbk = \mathbb{Z}$, we see that $\Bbbk \otimes_\mathbb{Z} P_Z(\mathbb{Z})$ is perverse. By formula (2.6.7) in [KS], we see that for any $\mathcal{A} \in \mathrm{P}_{G_\mathcal{O}}(Z, \Bbbk)$ we have

$$(10.9)\quad \operatorname{Hom}_\Bbbk(\Bbbk \otimes_\mathbb{Z} P_Z(\mathbb{Z}), \mathcal{A}) \;=\; \operatorname{Hom}_\mathbb{Z}(P_Z(\mathbb{Z}), \mathrm{R}\,\mathit{Hom}(\Bbbk, \mathcal{A})) \;=\; \operatorname{Hom}_\mathbb{Z}(P_Z(\mathbb{Z}), \mathcal{A}) \;=\; \mathrm{H}^*(\mathcal{G}r, \mathcal{A}) .$$

Thus, $\Bbbk \otimes_\mathbb{Z} P_Z(\mathbb{Z})$ represents the functor F on $\mathrm{P}_{G_\mathcal{O}}(Z, \Bbbk)$ and hence we must have $\Bbbk \otimes_\mathbb{Z} P_Z(\mathbb{Z}) = P_Z(\Bbbk)$.

Finally to get statement (b) for an arbitrary ring $\Bbbk$, it suffices to use (c), (b) for the case $\Bbbk = \mathbb{Z}$, and the fact that $\Bbbk \overset{L}{\underset{\mathbb{Z}}{\otimes}} \mathcal{J}_!(\lambda, \mathbb{Z}) \cong \mathcal{J}_!(\lambda, \Bbbk)$.

□

Let us write $\mathrm{P}_{G_{\mathcal{O}}}^{\Bbbk-proj}(Z, \Bbbk)$ for the subcategory of $\mathrm{P}_{G_{\mathcal{O}}}(Z, \Bbbk)$ consisting of sheaves $\mathcal{A} \in \mathrm{P}_{G_{\mathcal{O}}}(Z, \Bbbk)$ such that $\mathrm{H}^*(\mathcal{G}r, \mathcal{A})$ is $\Bbbk$-projective. Note that, by (3.9), the category $\mathrm{P}_{G_{\mathcal{O}}}^{\Bbbk-proj}(Z, \Bbbk)$ is closed under Verdier duality. Because $P_Z(\Bbbk) \in \mathrm{P}_{G_{\mathcal{O}}}^{\Bbbk-proj}(Z, \Bbbk)$, its dual $I_Z(\Bbbk) = \mathbb{D}(P_Z(\Bbbk))$ also belongs in $\mathrm{P}_{G_{\mathcal{O}}}^{\Bbbk-proj}(Z, \Bbbk)$. The sheaf $I_Z(\Bbbk)$ is an injective object in the subcategory $\mathrm{P}_{G_{\mathcal{O}}}^{\Bbbk-proj}(Z, \Bbbk)$. Note also that the abelianization of $\mathrm{P}_{G_{\mathcal{O}}}^{\Bbbk-proj}(\mathcal{G}r, \Bbbk)$ is $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}r, \Bbbk)$.

11. **Construction of the group scheme**

In this section we construct a group scheme $\tilde{G}_\Bbbk$ such that $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}r, \Bbbk)$ is the category of its representations. We proceed by Tannakian formalism, see, for example, [DM]. Unlike [DM] we work over an arbitrary commutative ring $\Bbbk$. This is made possible by the fact that $F(P_Z(\Bbbk))$ is free over $\Bbbk$, 10.1.

11.1. **Proposition.** *There is a group scheme $\tilde{G}_\Bbbk$ over $\Bbbk$ such that the tensor category of representations, finitely generated over $\Bbbk$, is equivalent to $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}r, \Bbbk)$. Furthermore, the coordinate ring $\Bbbk[\tilde{G}_\Bbbk]$ is free over $\Bbbk$ and $\tilde{G}_\Bbbk = \mathrm{Spec}(\Bbbk) \times_{\mathrm{Spec}(\mathbb{Z})} \tilde{G}_{\mathbb{Z}}$.*

Proof. We view $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}r, \Bbbk)$ as a direct limit $\varinjlim_Z \mathrm{P}_{G_{\mathcal{O}}}(Z, \Bbbk)$; here Z runs through finite dimensional $G_{\mathcal{O}}$-invariant closed subsets of the affine Grassmannian $\mathcal{G}r$. Let us write $A_Z(\Bbbk)$ for the $\Bbbk$-algebra $\mathrm{End}(P_Z(\Bbbk)) = F(P_Z(\Bbbk))$. The algebra $A_Z(\Bbbk)$ is free of finite rank over $\Bbbk$. Let us write $\mathrm{Mod}_{A_Z(\Bbbk)}$ for the category of $A_Z(\Bbbk)$-modules which are finitely generated over $\Bbbk$. Because $P_Z(\Bbbk)$ is a projective generator of $\mathrm{P}_{G_{\mathcal{O}}}(Z, \Bbbk)$, we see that the restriction of the functor F to $\mathrm{P}_{G_{\mathcal{O}}}(Z, \Bbbk) \to \mathrm{Mod}_\Bbbk$ lifts to an equivalence of abelian categories:

(11.1) the categories $\mathrm{P}_{G_{\mathcal{O}}}(Z, \Bbbk)$ and $\mathrm{Mod}_{A_Z(\Bbbk)}$ are equivalent as abelian categories.

As $A_Z(\Bbbk)$ is free over $\Bbbk$, its $\Bbbk$-dual $B_Z(\Bbbk)$ is naturally a co-algebra. Furthermore, it is equivalent to give a structure of an $A_Z(\Bbbk)$-module or a $B_Z(\Bbbk)$-comodule to a $\Bbbk$-module V:

$$\mathrm{Hom}_\Bbbk(A_Z(\Bbbk) \otimes_\Bbbk V, V) \;=\; \mathrm{Hom}_\Bbbk(V, B_Z(\Bbbk) \otimes_\Bbbk V)\,. \tag{11.2}$$

Thus we conclude:

(11.3)
the categories $\mathrm{P}_{G_{\mathcal{O}}}(Z, \Bbbk)$ and $\mathrm{Comod}_{B_Z(\Bbbk)}$ are equivalent as abelian categories.

Let us write $I_Z(\Bbbk)$ for the Verdier dual of $P_Z(\Bbbk)$. Now,

$$B_Z(\Bbbk) \;=\; A_Z(\Bbbk)^* \;=\; \mathrm{H}^*(\mathcal{G}r, P_Z(\Bbbk))^* \;=\; \mathrm{H}^*(\mathcal{G}r, I_Z(\Bbbk)) \;=\; F(I_Z(\Bbbk))\,. \tag{11.4}$$

If $Z \subset Z'$ are both closed and $G_{\mathcal{O}}$-invariant then the canonical morphism $P_{Z'}(\Bbbk) \to P_Z(\Bbbk)$ gives rise to a morphism $I_Z(\Bbbk) \to I_{Z'}(\Bbbk)$ and this, in turn, gives a map of co-algebras $B_Z(\Bbbk) \to B_{Z'}(\Bbbk)$. Hence we can form the coalgebra

$$B(\Bbbk) \;=\; \varinjlim B_Z(\Bbbk)\,, \tag{11.5}$$

and we get:

(11.6) the categories $\mathrm{P}_{G_\mathcal{O}}(\mathcal{G}\mathrm{r},\Bbbk)$ and $\mathrm{Comod}_{B(\Bbbk)}$ are equivalent as abelian categories.

It now remains to give the coalgebra $B(\Bbbk)$ the structure of an algebra and to give an inverse in its coalgebra structure. We will start by giving $B(\Bbbk)$ an algebra structure. To this end, let us consider the filtration of $\mathrm{P}_{G_\mathcal{O}}(\mathcal{G}\mathrm{r},\Bbbk)$ by the subcategories $\mathrm{P}_{G_\mathcal{O}}(\lambda,\Bbbk) = \mathrm{P}_{G_\mathcal{O}}(\overline{\mathcal{G}\mathrm{r}_\lambda},\Bbbk)$ indexed dominant coweights λ. This filtration is compatible with the convolution product in the sense that $\mathrm{P}_{G_\mathcal{O}}(\lambda,\Bbbk) * \mathrm{P}_{G_\mathcal{O}}(\mu,\Bbbk) \subseteq \mathrm{P}_{G_\mathcal{O}}(\lambda+\mu,\Bbbk)$. We have:

$$\begin{aligned}\mathrm{Hom}[P_{\lambda+\mu}(\Bbbk), P_\lambda(\Bbbk) * P_\mu(\Bbbk)] &\cong F[P_\lambda(\Bbbk) * P_\mu(\Bbbk)] \cong \\ &F[P_\lambda(\Bbbk)]\underset{\Bbbk}{\otimes} F[P_\mu(\Bbbk)] = A_\lambda(\Bbbk)\otimes_\Bbbk A_\mu(\Bbbk)\,.\end{aligned} \tag{11.7}$$

The element $1\otimes 1 \in A_\lambda(\Bbbk)\otimes_\Bbbk A_\lambda(\Bbbk)$ gives rise to a morphism $P_{\lambda+\mu}(\Bbbk) \to P_\lambda(\Bbbk) * P_\mu(\Bbbk)$. Dualizing this gives a morphism $I_\lambda(\Bbbk) * I_\mu(\Bbbk) \to I_{\lambda+\mu}(\Bbbk)$ and by applying the functor F a morphism

$$B_\lambda(\Bbbk)\otimes_\Bbbk B_\mu(\Bbbk) \to B_{\lambda+\mu}(\Bbbk)\,. \tag{11.8}$$

Passing to the limit gives $B(\Bbbk)$ a structure of a commutative $\Bbbk$-algebra; the associativity and the commutativity of the multiplication come from the associativity and commutativity of the tensor product. To summarize, we have constructed an affine monoid $\tilde{G}_\Bbbk = \mathrm{Spec}(B(\Bbbk))$ such that

$$\mathrm{Rep}_{\tilde{G}_\Bbbk} \quad \text{is equivalent to} \quad \mathrm{P}_{G_\mathcal{O}}(\mathcal{G}\mathrm{r},\Bbbk) \quad \text{as tensor categories}\,; \tag{11.9}$$

here $\mathrm{Rep}_{\tilde{G}_\Bbbk}$ denotes the category of representations of $\tilde{G}_\Bbbk$ which are finitely generated as $\Bbbk$-modules.

We will show next that $\tilde{G}_\Bbbk$ is a group scheme. To this end, we define an involution on $\mathrm{P}^{\Bbbk-proj}_{G_\mathcal{O}}(\mathcal{G}\mathrm{r},\Bbbk)$ as follows. Since the pro-algebraic group $G_\mathcal{O}$ acts freely on the ind-scheme $G_\mathcal{K}$ on the right we have the equivalence

$$\mathrm{P}_{G_\mathcal{O}}(\mathcal{G}\mathrm{r},\Bbbk) = \; \mathrm{P}_{G_\mathcal{O}}(G_\mathcal{K}/G_\mathcal{O},\Bbbk) \cong \; \mathrm{P}_{G_\mathcal{O}\times G_\mathcal{O}}(G_\mathcal{K},\Bbbk)\,. \tag{11.10}$$

On $G_\mathcal{K}$ we can consider the map $i: G_\mathcal{K}\to G_\mathcal{K}$, $i(g) = g^{-1}$. Using the above equivalence this gives us a functor $i^*: \mathrm{P}_{G_\mathcal{O}}(\mathcal{G}\mathrm{r},\Bbbk) \to \mathrm{P}_{G_\mathcal{O}}(\mathcal{G}\mathrm{r},\Bbbk)$. The involution on $\mathrm{P}^{\Bbbk-proj}_{G_\mathcal{O}}(\mathcal{G}\mathrm{r},\Bbbk)$ is now given by

$$\mathcal{A} \mapsto \mathcal{A}^* = \mathbb{D}(i^*\mathcal{A})\,. \tag{11.11}$$

This involution makes $\mathrm{Rep}^{\Bbbk-proj}_{\tilde{G}_\Bbbk}$, the category of representations of $\tilde{G}_\Bbbk$ on finitely generated projective $\Bbbk$-modules, a rigid tensor category. By [Sa] II.3.1.1, I.5.2.2, and I.5.2.3, we conclude that $\tilde{G}_\Bbbk = \mathrm{Spec}(B(\Bbbk))$ is a group scheme.

The statement $\tilde{G}_\Bbbk = \mathrm{Spec}(\Bbbk)\times_{\mathrm{Spec}(\mathbb{Z})}\tilde{G}_\mathbb{Z}$, i.e., that $B(\Bbbk) = \Bbbk\otimes_\mathbb{Z} B(\mathbb{Z})$, now follows from Proposition 10.1 part (c). The algebra $B(\Bbbk)$ is free over $\Bbbk$ by construction.

12. The identification of $\tilde{G}_{\Bbbk}$ with the dual group of G

In this section we identify the group scheme $\tilde{G}_{\Bbbk}$. As $\tilde{G}_{\Bbbk} = \operatorname{Spec}(\Bbbk) \times_{\operatorname{Spec}(\mathbb{Z})} \tilde{G}_{\mathbb{Z}}$, by 11.1, it suffices to do so when $\Bbbk = \mathbb{Z}$. Recall that there exists a unique split reductive group scheme, the Chevalley group scheme, over $\mathbb{Z}$ associated to any root datum. In this section we will show that $\tilde{G}_{\mathbb{Z}}$ is the Chevalley scheme associated to the root datum of the dual group $\check{G}$:

12.1. **Theorem.** *The group scheme $\tilde{G}_{\mathbb{Z}}$ is the split reductive group scheme over $\mathbb{Z}$ whose root datum is dual to that of G.*

Recall that we have shown in section 7 that the above statement holds at the generic point, i.e., that $\tilde{G}_{\mathbb{Q}}$ is split reductive reductive group whose root datum is dual to that of G. By proposition 11.1 the group scheme $\tilde{G}_{\mathbb{Z}}$ is flat over $\operatorname{Spec}(\mathbb{Z})$. Note, however, that we do not yet know that the group scheme $\operatorname{Spec}(\mathbb{Z})$ is of finite type. This will slightly complicate some of the arguments in this section. We first observe that by the uniqueness of the Chevalley group scheme, [Dem], and the fact that $\tilde{G}_{\mathbb{Q}}$ is split reductive reductive group whose root datum is dual to that of G it suffices to show:

(12.1a) The group scheme $\tilde{G}_{\mathbb{Z}}$ is smooth over $\operatorname{Spec}(\mathbb{Z})$

(12.1b) The dual (split) torus $\check{T}_{\mathbb{Z}}$ is a maximal torus of $\tilde{G}_{\mathbb{Z}}$.

(12.1c) At each geometric point $\operatorname{Spec}(\kappa)$, $\kappa = \bar{\mathbb{F}}_p$, the group scheme $\tilde{G}_\kappa$ is reductive.

Note that it is not necessary to check in (12.1c) that $\tilde{G}_\kappa$ has root datum dual to that of G; that is automatic. However, to prove the fact that $\tilde{G}_{\mathbb{Z}}$ is smooth over $\operatorname{Spec}(\mathbb{Z})$ and to deal with the fact that we do not yet know that $\tilde{G}_{\mathbb{Z}}$ is of finite type we end up having to calculate the root data of the $\tilde{G}_\kappa$.

We begin by exhibiting $\check{T}_{\mathbb{Z}}$ as a subtorus of $\tilde{G}_{\mathbb{Z}}$. Proceeding just as in section 7, the functor

$$F = \mathrm{H}^* : \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}, \Bbbk) \to \operatorname{Mod}_{\mathbb{Z}}(X_*(T))) \tag{12.2}$$

gives us a homomorphism $\check{T}_{\mathbb{Z}} \to \tilde{G}_{\mathbb{Z}}$. On the other hand, by theorem 7.3 we know (12.1b) when $\kappa = \bar{\mathbb{Q}}$. By arguing as in section 7 we see that for $\kappa = \bar{\mathbb{F}}_p$ the map $\check{T}_\kappa \to \tilde{G}_\kappa$ is an embedding and that the group scheme $\tilde{G}_\kappa$ is connected.

Next, let us write $(\tilde{G}_\kappa)_{red}$ for the reduced subgroup scheme of $\tilde{G}_\kappa$. From the discussion above and by appealing to theorem 7.2 of [PY] we conclude that it suffices to show that:

(12.3) For $\kappa = \bar{\mathbb{F}}_p$ the torus $\check{T}_\kappa$ is maximal in $(\tilde{G}_\kappa)_{red}$ and the group scheme $(\tilde{G}_\kappa)_{red}$ is reductive with root datum dual to that of G.

Note that, except in characteristic 2 where it is necessary in any case, the statement that $(\tilde{G}_\kappa)_{red}$ is reductive with root datum dual to that of G is only needed to guarantee that all the fibers $\tilde{G}_\kappa$ are of the same dimension. Before proving the above statement, we make one more argument using the entire flat family and after that we only work at $\kappa = \bar{\mathbb{F}}_p$. The flatness of $\tilde{G}_{\mathbb{Z}}$ implies that

$$\dim G = \dim \tilde{G}_{\mathbb{Q}} \geq \dim(\tilde{G}_\kappa)_{red}\,, \quad \text{where} \quad \kappa = \bar{\mathbb{F}}_p\,. \tag{12.4}$$

To see this, it suffices to observe that the lifts of algebraically independent elements in the coordinate ring of $(\tilde{G}_{\mathbb{F}_p})_{red}$ are algebraically independent in $\mathbb{Q}[\tilde{G}_{\mathbb{Q}}]$. Note that we do not a priori have an equality in (12.4) as we do not know that $\tilde{G}_{\mathbb{Z}}$ is of finite type.

For any group scheme H let us write Irr_H for the set of irreducible representations of H. We choose a quotient group scheme G^*_κ of $\tilde{G}_\kappa$ with the following properties:

$$G^*_\kappa \text{ is of finite type} \tag{12.5a}$$

$$\text{the canonical map } \mathrm{Irr}_{G^*_\kappa} \to \mathrm{Irr}_{\tilde{G}_\kappa} \text{ is a bijection} \tag{12.5b}$$

$$(G^*_\kappa)_{red} = (\tilde{G}_\kappa)_{red} \tag{12.5c}$$

It is possible to satisfy the first and third conditions because any group scheme is a projective limit of group schemes of finite type and by (12.4) the $(\tilde{G}_\kappa)_{red}$ is of finite type. To see that (12.5b) can be satisfied it suffices to note that it is enough to choose G^*_κ sufficiently large so that the irreducible representations $L^{\tilde{G}}(\omega_i)$ associated to the fundamental weights ω_i are pull-backs of representations of G^*_κ. Then, as for any dominant coweights λ, μ, the support of the convolution product $\mathcal{I}_{!*}(\lambda) * \mathcal{I}_{!*}(\mu)$ is $\overline{\mathcal{G}\mathrm{r}_{\lambda+\mu}}$ we see that $\mathcal{I}_{!*}(\lambda+\mu)$ has to be a subquotient of the convolution product $\mathcal{I}_{!*}(\lambda) * \mathcal{I}_{!*}(\mu)$. Therefore, all irreducible representations $L^{\tilde{G}}(\lambda)$ of $\tilde{G}_\kappa$ come from G^*_κ.

Let us write R for the reductive quotient of $(\tilde{G}_\kappa)_{red} = (G^*_\kappa)_{red}$. As any irreducible representation of $(G^*_\kappa)_{red}$ is trivial on the unipotent radical we have the equality

$$\mathrm{Irr}_{(G^*_\kappa)_{red}} = \mathrm{Irr}_R\ . \tag{12.6}$$

Furthermore, note that the torus $\check{T}_\kappa \subset (G^*_\kappa)_{red}$ is also a subtorus of R. We now argue that in order to prove (12.3) it suffices to show that:

$$\begin{aligned} &\text{the torus } \check{T}_\kappa \text{ is maximal in } R \text{ and the root data of } R \\ &\text{with respect to } \check{T}_\kappa \text{ and } G \text{ with respect to } T \text{ are dual}. \end{aligned} \tag{12.7}$$

The above statement implies that $\dim(R) = \dim(G)$, and, together with (12.4), this gives

$$\dim G = \dim \tilde{G}_{\mathbb{Q}} \geq \dim(G^*_\kappa)_{red} \geq \dim R = \dim G\,. \tag{12.8}$$

Now $(G^*_\kappa)_{red} = R$ and so $(G^*_\kappa)_{red}$ is reductive with its root datum dual to that of G.

To argue (12.7) we make the following construction. Let us write

$$\begin{array}{ccc} G^*_\kappa & \xrightarrow{\mathrm{Fr}^n_{G^*_\kappa}} & (G^*_\kappa)^{(n)} \\ \uparrow & & \uparrow \\ (G^*_\kappa)_{red} & \xrightarrow{\mathrm{Fr}^n_{G^*_\kappa}} & (G^*_\kappa)^{(n)}_{red} \end{array} \tag{12.9}$$

for the nth Frobenius twists. By [DG, corollary III.3.6.4] we see that as a scheme with the right multiplication action of $(G^*_\kappa)_{red}$, $G^*_\kappa \cong (G^*_\kappa/(G^*_\kappa)_{red})\times(G^*_\kappa)_{red}$ and that the coordinate ring of $G^*_\kappa/(G^*_\kappa)_{red}$ is of the form $\kappa[G^*_\kappa/(G^*_\kappa)_{red}] = \kappa[X_1, ..., X_n]/\langle X_1^{p_1}, ..., X_n^{p_n}\rangle$ for some powers $p_i = p^{e_i}$ of p. Hence, for $n \geq e_i$,

$$\text{the Frobenius map } \mathrm{Fr}^n : G^*_\kappa \to (G^*_\kappa)^{(n)} \text{ factors through } (G^*_\kappa)^{(n)}_{red}. \tag{12.10}$$

This implies, first of all, that

$$\begin{gathered}\text{the n}^{th}\text{ Frobenius twists of irreducible representations}\\ \text{of } (G^*_\kappa)_{red} \text{ lift to irreducible representations of } G^*_\kappa.\end{gathered} \tag{12.11}$$

The irreducible representations of $(G^*_\kappa)_{red}$ are parametrized by Weyl group orbits of characters of a maximal torus S in $(G^*_\kappa)_{red}$ and this maximal torus S can be chosen to contain $\check{T}_\kappa$. This immediately implies that $\check{T}_\kappa$ must be a maximal torus of $(G^*_\kappa)_{red}$.

We will now argue the second part of (12.7), i.e., that

$$\Phi(R, \check{T}_\kappa) = \check{\Phi}(G, T) \quad \text{and} \quad \check{\Phi}(R, \check{T}_\kappa) = \Phi(G, T). \tag{12.12}$$

The argument here is a bit more involved than the argument in characteristic zero in section 7, but the basic idea is the same: the pattern of the weights of irreducible representations determines the root datum (at least almost!).

First of all, from the previous discussion we conclude that

$$L^{\tilde{G}}(p^n\lambda) \ = \ L^R(p^n\lambda) \quad \text{for all } \lambda \in X_*(T). \tag{12.13}$$

As the irreducible representations of $\tilde{G}_\kappa$ are parametrized by Weyl group orbits in $X_*(T)$, this implies immediately that the walls of the Weyl chambers of the root systems of (G, T) and $(R, \check{T}_\kappa)$ coincide in $X_*(T)$. Furthermore, from the patterns of the weights of the representations in (12.13) we conclude that the simple root directions of R coincide with the simple coroot directions of G. Recall that in characteristic zero we got the equality of simple roots of R and the simple coroots of G, but we cannot immediately conclude this fact here. Hence, we have to argue slightly differently.

The center of a reductive group R is the group scheme $\mathrm{Hom}(X^*(\check{T}_\kappa)/\mathbb{Z}{\cdot}\Phi(R, \check{T}_\kappa), \mathbb{G}_{m,\kappa})$. On the other hand, the tensor category $\mathcal{P}_{G_{\mathcal{O}}}(\mathcal{G}r, \Bbbk)$ is graded by the abelian group $\pi_0(\mathcal{G}r) \cong \pi_1(G) = X_*(T)/\mathbb{Z}{\cdot}\check{\Phi}(G, T)$. This grading exhibits the group

$$\mathrm{Hom}(X_*(T)/\mathbb{Z}{\cdot}\check{\Phi}(G, T), \mathbb{G}_{m,\kappa}) \tag{12.14}$$

as subgroup scheme of the center of $\tilde{G}_\kappa$ and hence as an algebraic subgroup of the center of R. Hence,

$$\mathrm{Hom}(X_*(T)/\mathbb{Z}{\cdot}\check{\Phi}(G, T), \mathbb{G}_{m,\kappa}) \ \subset \ \mathrm{Hom}(X^*(\check{T}_\kappa)/\mathbb{Z}{\cdot}\Phi(R, \check{T}_\kappa), \mathbb{G}_{m,\kappa}). \tag{12.15}$$

Thus, we get a surjection

$$X^*(\check{T}_\kappa)/\mathbb{Z}{\cdot}\Phi(R, \check{T}_\kappa) \quad \twoheadrightarrow \quad X_*(T)/\mathbb{Z}{\cdot}\check{\Phi}(G, T). \tag{12.16}$$

This implies that

$$\mathbb{Z}{\cdot}\Phi(R, \check{T}_\kappa) \ \subset \ \mathbb{Z}{\cdot}\check{\Phi}(G, T) \tag{12.17a}$$

and then we get dually, using the fact $\langle\check{\alpha},\alpha\rangle = 2$ that

(12.17b) $$\mathbb{Z}{\cdot}\Phi(G,T) \;\subset\; \mathbb{Z}{\cdot}\check{\Phi}(R,\check{T}_\kappa)\,.$$

If G if adjoint type, then $\mathbb{Z}{\cdot}\Phi(G,T) = X^*(T)$ and hence $\mathbb{Z}{\cdot}\Phi(G,T) = \mathbb{Z}{\cdot}\check{\Phi}(R,\check{T}_\kappa)$. This, then, gives (12.12) and concludes the proof in case G is of adjoint type.

We assume next that G is semi-simple and write

(12.18) $$G \twoheadrightarrow G_{\mathrm{ad}}\,, \quad \text{where } G_{\mathrm{ad}} \text{ is the adjoint quotient of } G\,.$$

We have already shown that $(\tilde{G}_{\mathrm{ad}})_{\mathbb{Z}} \cong (\check{G}_{\mathrm{ad}})_{\mathbb{Z}}$. Now $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}r_G,\Bbbk)$ is a tensor subcategory of $\mathrm{P}_{G_{\mathrm{ad},\mathcal{O}}}(\mathcal{G}r_{G_{\mathrm{ad}}},\Bbbk)$ and this gives us a surjective homomorphism

(12.19) $$(\check{G}_{\mathrm{ad}})_{\mathbb{Z}} \twoheadrightarrow \tilde{G}_{\mathbb{Z}}\,.$$

This implies that all the fibers $\tilde{G}_{\bar{\mathbb{F}}_p}$ are reductive. Thus, by the uniqueness of the Chevalley group scheme, it suffices to show that $\tilde{G}_{\mathbb{Z}}$ is of finite type. We know, by theorem 7.2 of [PY] that $\tilde{G}_{\mathbb{Z}}$ is of finite type outside of the prime 2. As $\mathbb{Z}[\tilde{G}_{\mathbb{Z}}]$ is free over $\mathbb{Z}$, we can proceed as in (12.5) and section 5.1.4 of [PY] and exhibit $\tilde{G}_{\mathbb{Z}}$ as an inverse limit of group schemes $\tilde{G}^i_{\mathbb{Z}}$ such that

(12.20a) the group schemes $\tilde{G}^i_{\mathbb{Z}}$ are of finite type over $\mathbb{Z}$

(12.20b) outside of the prime 2 the group schemes $\tilde{G}^i_{\mathbb{Z}}$ and $\tilde{G}_{\mathbb{Z}}$ coincide

(12.20c) $$\tilde{G}^i_{\bar{\mathbb{F}}_2} \;=\; \tilde{G}_{\bar{\mathbb{F}}_2}\,.$$

The properties above imply that $\tilde{G}^i_{\mathbb{Z}}$ is the Chevalley scheme $\check{G}_{\mathbb{Z}}$. By uniqueness all the $\tilde{G}^i_{\mathbb{Z}}$ coincide and hence $\tilde{G}^i_{\mathbb{Z}} = \check{G}_{\mathbb{Z}}$. This concludes the proof in the semi-simple case.

Let us now consider the case of a general reductive G. Let us write $S = Z(G)^0$ for the connected component of the center of G. Then we have an exact sequence

(12.21) $$1 \to S \to G \to G_{\mathrm{der}} \to 1\,,$$

where G_{der} is the derived group of G. This gives the following diagram of maps:

(12.22) $$\mathcal{G}r_S \xrightarrow{\;\;i\;\;} \mathcal{G}r_G \xrightarrow{\;\;\pi\;\;} \mathcal{G}r_{G_{\mathrm{der}}}\,,$$

which exhibits $\mathcal{G}r_G$ as a trivial cover of $\mathcal{G}r_{G_{\mathrm{der}}}$ with fiber $\mathcal{G}r_S$. By taking pushfowards of sheaves this gives us the following sequence of functors:

(12.23) $$\mathrm{P}_{S_{\mathcal{O}}}(\mathcal{G}r_S,\mathbb{Z}) \xrightarrow{\;\;\omega\;\;} \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}r_G,\mathbb{Z}) \xrightarrow{\;\;\gamma\;\;} \mathrm{P}_{G_{\mathrm{der}\mathcal{O}}}(\mathcal{G}r_{G_{\mathrm{der}}},\mathbb{Z})\,,$$

whereω is clearly an embedding and ω is essentially surjective because of the triviality of the cover. This, in turn, gives the following exact sequence of group schemes:

(12.24) $$1 \to (\check{G}_{\mathrm{der}})_{\mathbb{Z}} \to \tilde{G}_{\mathbb{Z}} \to \check{S}_{\mathbb{Z}} \to 1$$

The fact that we have exactness at both ends follows from the fact ω is an embedding and γ is essentially surjective. To see the exactness in the middle, let us consider the quotient $\tilde{G}_{\mathbb{Z}}/(\check{G}_{\mathrm{der}})_{\mathbb{Z}}$. The representations of this groups scheme are given by objects in $\mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}r_G,\mathbb{Z})$ whose push-forward under π to $\mathcal{G}r_{G_{\mathrm{der}}}$ consists of direct sums of trivial representations. But these constitute precisely $\mathrm{P}_{S_{\mathcal{O}}}(\mathcal{G}r_S,\mathbb{Z})$. Hence, $\tilde{G}_{\mathbb{Z}}$ is smooth and of finite type. This, finally, concludes the proof.

13. **Representations of reductive groups**

The point of view we have taken in this paper so far is that of giving a canonical, geometric construction of the dual group. In this section we turn things around and view our work as giving a geometric interpretation of representation theory of split reductive groups.

As before, $\mathbb{k}$ is commutative ring, noetherian and of finite global dimension. Recall the content of our main theorem 12.1:

$$\operatorname{Rep}_{\check{G}_{\mathbb{k}}} \text{ is equivalent to } \mathrm{P}_{G_{\mathcal{O}}}(\mathcal{G}\mathrm{r}, \mathbb{k}) \text{ as tensor categories;} \tag{13.1}$$

here $\operatorname{Rep}_{\check{G}_{\mathbb{k}}}$ stand for the category of $\mathbb{k}$-representations of $\check{G}_{\mathbb{k}}$, finitely generated over $\mathbb{k}$, and $\check{G}_{\mathbb{k}}$ stands for the canonical split group scheme associated to the root datum dual to that of the complex group G. This way we get a geometric interpretation of representation theory of $\check{G}_{\mathbb{k}}$. The case when $\mathrm{Char}(\mathbb{k}) = 0$ was discussed in section 7.

Following our previous discussion we have $\check{T}_{\mathbb{k}} \subset \check{B}_{\mathbb{k}} \subset \check{G}_{\mathbb{k}}$, a maximal torus and a Borel in $\check{G}_{\mathbb{k}}$. Associated to a weight $\lambda \in X_*(\check{T})$ there are two standard representations of highest weight λ. Let us describe these representations. We extend λ to a character on $\check{B}_{\mathbb{k}}$ so that it is trivial on the unipotent radical of $\check{B}_{\mathbb{k}}$ and then induce this character to a representation of $\check{G}_{\mathbb{k}}$. We call the resulting representation the Schur module and denote it by $S(\lambda)$. As a module it is free over $\mathbb{k}$. The other representation associated to λ is the Weyl module $W(\lambda) = S(-w_0\lambda)^*$, where w_0 is the longest element in the Weyl group. There is a canonical morphism $W(\lambda) \to S(\lambda)$ which is the identity on the λ-weight space. We have:

13.1. **Proposition.** *Under the equivalence* (13.1) *the diagrams $W(\lambda) \to S(\lambda)$ and $\mathcal{I}_!(\lambda) \to \mathcal{I}_*(\lambda)$ correspond to each other.*

Proof. The modules $S(\lambda)$ and $W(\lambda)$ can also be characterized in the following manner. Let us write $\operatorname{Rep}^{\leq\lambda}_{\check{G}_{\mathbb{k}}}$ for the full subcategory of $\operatorname{Rep}_{\check{G}_{\mathbb{k}}}$ consisting of representations whose $\check{T}_{\mathbb{k}}$-weights are all $\leq \lambda$. Then the representations $S(\lambda)$ and $W(\lambda)$ satisfy the following universal properties:

$$\text{for } V \in \operatorname{Rep}^{\leq\lambda}_{\check{G}_{\mathbb{k}}} \quad \text{we have} \quad \operatorname{Hom}_{\check{G}_{\mathbb{k}}}(V, S(\lambda)) = \operatorname{Hom}_{\mathbb{k}}(V_\lambda, \mathbb{k}) \tag{13.2a}$$

and

$$\text{for } V \in \operatorname{Rep}^{\leq\lambda}_{\check{G}_{\mathbb{k}}} \quad \text{we have} \quad \operatorname{Hom}_{\check{G}_{\mathbb{k}}}(W(\lambda), V) = \operatorname{Hom}_{\mathbb{k}}(\mathbb{k}, V_\lambda) \tag{13.2b}$$

On the geometric side the category $\operatorname{Rep}^{\leq\lambda}_{\check{G}_{\mathbb{k}}}$ corresponds to the category $\mathrm{P}_{G_{\mathcal{O}}}(\lambda, \mathbb{k}) = \mathrm{P}_{G_{\mathcal{O}}}(\overline{\mathcal{G}\mathrm{r}_\lambda}, \mathbb{k})$. Obviously, the sheaves $\mathcal{I}_*(\lambda)$ and $\mathcal{I}_!(\lambda)$ belong $\mathrm{P}_{G_{\mathcal{O}}}(\lambda, \mathbb{k})$ and satisfy the universal properties (13.2), proving the proposition. □

As a corollary, proposition 3.10 gives:

13.2. **Corollary.** *The ν-weight spaces $S(\lambda)_\nu$ and $W(\lambda)_\nu$ of $S(\lambda)$ and $W(\lambda)$, respectively, can both be canonically identified with the $\mathbb{k}$-vector space spanned by the irreducible components of $\overline{\mathcal{G}r_\lambda} \cap S_\nu$. In particular, the dimensions of these weight spaces are given by the number of irreducible components of $\overline{\mathcal{G}r_\lambda} \cap S_\nu$.*

Finally, let us assume that $\Bbbk$ is a field. Then we also have an irreducible representation $L(\lambda)$ associated to λ. Under the correspondence (13.1) the irreducible representation $L(\lambda)$ corresponds to the irreducible sheaf $\mathcal{J}_{!*}(\lambda, \Bbbk)$.

14. Variants and the geometric Langlands program

As was stated before, in this paper we have worked with $\mathbb{C}$-schemes because in this case we have a good sheaf theory for sheaves with coefficients in any commutative ring, in particular, the integers. It is also possible to work with other topologies. This is important for certain applications, for example for the geometric Langlands program, since our results can be viewed as providing the unramified local geometric Langlands correspondence.

We will explain briefly the modifications necessary to work in the etale topology and over an arbitrary algebraically closed base field K. To this end, let us view the group G as a split reductive group over the integers. All the geometric constructions made in this paper go through over the integers, in particular, our Grassmannian $\mathcal{G}\mathrm{r}$ is defined over $\mathbb{Z}$. We write $\mathcal{G}\mathrm{r}_K$ for the affine Grassmannian over the base field K. In a few places in the paper we have argued using $\mathbb{Z}$ as coefficients, for instance, in section 12. When we work in the etale topology, we simply replace $\mathbb{Z}$ by $\mathbb{Z}_\ell$, where $\ell \neq Char(K)$.

For completeness, we state here a version of our theorem for $\mathcal{G}\mathrm{r}_K$:

14.1. **Theorem.** *There is an equivalence of tensor categories*

$$\mathrm{P}_{G(\mathcal{O})}(\mathcal{G}r_K, \Bbbk) \cong Rep(\check{G}_{\Bbbk}).$$

where we can take $\Bbbk$ to be any ring for which the left hand side is defined and which can be obtained by base change from $\mathbb{Z}_\ell$, for example, $\Bbbk$ could be $\mathbb{Q}_\ell$, $\mathbb{Z}_\ell$, $\mathbb{Z}/\ell^n\mathbb{Z}$, $\bar{\mathbb{F}}_\ell$.

14.2. **Remark.** *The previous theorem can be used in the geometric Langlands program to define the notion of Hecke eigensheaves of automorphic sheaves with $\Bbbk$-coefficients.*

Appendix A. Categories of perverse sheaves

In this appendix we prove propositions 2.1 and 2.2, i.e., we will show that

A.1. **Proposition.** *The categories $\mathrm{P}_{G_\mathcal{O} \rtimes \mathrm{Aut}(\mathcal{O})}(\mathcal{G}r, \Bbbk)$, $\mathrm{P}_{G_\mathcal{O}}(\mathcal{G}r, \Bbbk)$, and $\mathrm{P}_\mathcal{S}(\mathcal{G}r, \Bbbk)$ are naturally equivalent.*

We first note that $\mathrm{P}_{G_\mathcal{O} \rtimes \mathrm{Aut}(\mathcal{O})}(\mathcal{G}\mathrm{r}, \Bbbk)$ is a full subcategory of $\mathrm{P}_{G_\mathcal{O}}(\mathcal{G}\mathrm{r}, \Bbbk)$ and $\mathrm{P}_{G_\mathcal{O}}(\mathcal{G}\mathrm{r}, \Bbbk)$ is a full subcategory of $\mathrm{P}_\mathcal{S}(\mathcal{G}\mathrm{r}, \Bbbk)$; this follows from the fact that stabilizers of points are connected for the actions of $G_\mathcal{O} \rtimes \mathrm{Aut}(\mathcal{O})$ and $G_\mathcal{O}$ on $\mathcal{G}\mathrm{r}$. The reductive quotients of $G_\mathcal{O} \rtimes \mathrm{Aut}(\mathcal{O})$, $G_\mathcal{O}$, and $\mathrm{Aut}(\mathcal{O})$ are $G \times \mathbb{G}_m$, G, and $\mathbb{G}_m$, respectively. Hence, it suffices to show that for any $G_\mathcal{O}$-invariant finite dimensional subvariety we have:

$$\text{(A.1)} \qquad \text{the categories } \mathrm{P}_{\mathcal{S},G}(Z, \Bbbk) \text{ and } \mathrm{P}_{\mathcal{S},\mathbb{G}_m}(Z, \Bbbk) \text{ are equivalent to } \mathrm{P}_\mathcal{S}(Z, \Bbbk)\,;$$

here $\mathrm{P}_{\mathcal{S},G}(Z, \Bbbk)$ and $\mathrm{P}_{\mathcal{S},\mathbb{G}_m}(Z, \Bbbk)$ denote subcategories of $\mathrm{P}_\mathcal{S}(Z, \Bbbk)$ consisting of sheaves which are G and $\mathbb{G}_m$- equivariant, respectively.

We will proceed by induction in the following manner. Obviously, the statement above holds if Z is a $G_{\mathcal{O}}$-orbit. Hence, by induction, it suffices to prove (A.1) for a $G_{\mathcal{O}}$-invariant subset Z under the following hypotheses:

$$\begin{gathered}\text{for some dominant } \lambda, \text{ the orbit } \mathcal{G}\mathrm{r}^\lambda \text{ is closed in } Z \\ \text{and (A.1) holds for the open set } U = Z - \mathcal{G}\mathrm{r}^\lambda .\end{gathered} \tag{A.2}$$

To prove the above statement, we use the gluing construction of [MV], [V] for perverse sheaves. Let us recall the construction. We write $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories and $F_1, G_1 : \mathcal{A} \to \mathcal{B}$ two functors, F_1 right exact, F_2 left exact, and $T : F_1 \to F_2$ a natural transformation. We define a category $\mathcal{C}(F_1, F_2; T)$ as follows. The objects of $\mathcal{C}(F_1, F_2; T)$ consist of pairs of objects $(A, B) \in \mathrm{Ob}(\mathcal{A}) \times \mathrm{Ob}(\mathcal{B})$ together with a factorization $F_1(A) \xrightarrow{m} B \xrightarrow{n} F_2(A)$ of $T(A)$, i.e., $n \circ m = T(A)$. The morphisms of $\mathcal{C}(F_1, F_2; T)$ are given by pairs of morphisms $(f, g) \in \mathrm{Mor}(\mathcal{A}) \times \mathrm{Mor}(\mathcal{B})$ which make the appropriate prism commute. The category $\mathcal{C}(F_1, F_2; T)$ is abelian.

We use this formalism in various situations. To begin with, let us write $j : U \hookrightarrow Z$ for the inclusion and set:

$$\begin{aligned} \mathcal{A} &= \mathrm{P}_{\mathcal{S}}(U, \Bbbk) \\ \mathcal{B} &= \mathrm{Mod}_{\Bbbk} \\ F_1 &= F_\lambda \circ {}^p j_! \\ F_2 &= F_\lambda \circ {}^p j_* \\ T &= F_\lambda({}^p j_! \to {}^p j_*) \,. \end{aligned} \tag{A.3}$$

We have a functor

$$E : \mathrm{P}_{\mathcal{S}}(Z, \Bbbk) \to \mathcal{C}(F_1, F_2; T) \tag{A.4}$$

which sends $\mathcal{F} \in \mathrm{P}_{\mathcal{S}}(Z, \Bbbk)$ to $A = \mathcal{F}\big|U$, $B = F_\lambda(\mathcal{F})$ and the factorization $F_1(A) \xrightarrow{m} B \xrightarrow{n} F_2(A)$ is the one gotten by applying F_λ to ${}^p j_!(\mathcal{F}\big|U) \to \mathcal{F} \to {}^p j_*(\mathcal{F}\big|U)$. By Proposition 1.2 in [V] the functor E is an embedding. Two remarks are in order. First, in [V] we work over a field, but this is not used in the proof. Secondly, the functor E is actually an equivalence of categories.

Let us now bring in the group G. We write $\tilde{\mathcal{S}}$ for the stratification of $G \times Z$ by subvarieties $G \times \mathcal{G}\mathrm{r}^\lambda$. We write $a : G \times Z \to Z$ for the action map and $p : G \times Z \to Z$ for the projection. Let $\mathcal{F} \in \mathrm{P}_{\mathcal{S}}(Z, \Bbbk)$. We have an isomorphism $\phi : p^*\mathcal{F}|G \times U \cong a^*\mathcal{F}|G \times U$ such that $\phi|\{e\} \times U = \mathrm{id}$. We are now to extend the ϕ to $G \times Z$. To this end we first construct a functor $\tilde{F}_\lambda : \mathrm{P}_{\tilde{\mathcal{S}}}(G \times Z, \Bbbk) \to \tilde{\mathcal{B}}$, where $\tilde{\mathcal{B}}$ stands for the category of $\Bbbk$-local systems on G. Let us write $\tilde{S}_\lambda = G \times S_\lambda$, denote by $i : \tilde{S}_\lambda \hookrightarrow G \times Z$ the inclusion, and write $\pi : G \times Z \to G$ for the projection. Then $\tilde{F}_\lambda = R\pi_! i^*$. Furthermore, we write $j : G \times U \hookrightarrow G \times G$ for the inclusion and set:

$$
\begin{aligned}
\tilde{\mathcal{A}} &= \mathrm{P}_{\mathcal{S}}(G \times U, \Bbbk)\\
\tilde{\mathcal{B}} &= \{\Bbbk\text{-local systems on } G\}\\
\tilde{F}_1 &= \tilde{F}_\lambda \circ {}^p\tilde{j}_!\\
\tilde{F}_2 &= \tilde{F}_\lambda \circ {}^p\tilde{j}_*\\
\tilde{T} &= \tilde{F}_\lambda({}^p\tilde{j}_! \to {}^p\tilde{j}_*)\,.
\end{aligned}
\tag{A.5}
$$

As before, we have a functor

$$
\tilde{E} : \mathrm{P}_{\tilde{\mathcal{S}}}(G \times Z, \Bbbk) \to \mathcal{C}(\tilde{F}_1, \tilde{F}_2; \tilde{T})
\tag{A.6}
$$

which sends $\tilde{\mathcal{F}} \in \mathrm{P}_{\mathcal{S}}(G \times Z, \Bbbk)$ to $\tilde{A} = \tilde{\mathcal{F}}\big|G \times U$, $\tilde{B} = \tilde{F}_\lambda(\tilde{\mathcal{F}})$ and the factorization $\tilde{F}_1(\tilde{A}) \xrightarrow{m} \tilde{B} \xrightarrow{n} \tilde{F}_2(\tilde{A})$ is the one gotten by applying $\tilde{F}_\lambda$ to ${}^p\tilde{j}_!(\tilde{\mathcal{F}}\big|G \times U) \to \tilde{\mathcal{F}} \to {}^p\tilde{j}_*(\tilde{\mathcal{F}}\big|G \times U)$. By the same reasoning as above, the functor $\tilde{E}$ is an equivalence of categories.

We will now apply $\tilde{E}$ to $p^*\mathcal{F}$ and to $a^*\mathcal{F}$. For $\tilde{E}(p^*\mathcal{F})$ we get the data of $E(\mathcal{F})$ at $\{e\} \times Z$ extended across G as the constant local system. For $\tilde{E}(a^*\mathcal{F})$ we also get the extension data of $E(\mathcal{F})$ at $\{e\} \times Z$. Because, by theorem 3.6, the functors F_ν are independent of the data $T \subset B$ used in defining them, we see that the extension data for $a^*\mathcal{F}$ restricted to $\{g\} \times Z$, for any $g \in G$, is canonically identified with the extension data of $a^*\mathcal{F}$ at $\{e\} \times Z$. This gives us an identification of $\tilde{E}(a^*\mathcal{F})$ with $\tilde{E}(p^*\mathcal{F})$ and hence an isomorphism between $a^*\mathcal{F}$ and $p^*\mathcal{F}$. This shows the first part of A.1.

The case of the group $\mathbb{G}_m$ is even a bit simpler. Here we use the fact that $\mathbb{G}_m$-action preserves the variety S_λ and hence all the $\mathbb{G}_m$-translates of the functor F_λ are identical to F_λ.

References

[BBD] A.Beilinson, J.Bernstein and P.Deligne, Faisceaux pervers, *Astérisque* **100** (1982).

[BD] A.Beilinson, V.Drinfeld, Quantization of hitchin integrable system and Hecke eigensheaves, preprint.

[BL] J.Bernstein, V.Lunts, Equivariant sheaves and functors, Lecture Notes in Math. **1578** (1994).

[BL1] A.Beauville and Y.Laszlo, Conformal blocks and generalized theta functions, *Comm. Math. Phys.* **164** (1994), 385 – 419.

[BL2] A.Beauville and Y.Laszlo, Un lemme de descent, *Comptes Rendus Acad. Sci. Paris* **320** Série I (1995) 335 – 340.

[Br] T.Braden, Hyperbolic localization of intersection cohomology, *Transformation Groups* **8** (2003), no. 3, 209–216.

[De] P.Deligne, Catégories tannakiennes, The Grothendieck Festschrift, **II**, 111–195, Progr. Math., 87, Birkhauser (1990).

[Dem] M.Demazure, Schémas en groupes réductifs, *Bull. Soc. Math. France* **93** (1965) 369–413.

[DG] M.Demazure, P.Gabriel, Groupes algbriques. Tome I: Géométrie algbrique, généralités, groupes commutatifs, Masson & Cie, Éditeur, Paris; North-Holland Publishing Co., Amsterdam (1970).

[DM] P.Deligne and J.Milne, Tannakian categories in “Hodge cycles and motives”, *Springer, Lecture notes* **900** (1982) 101 – 228.

[Gi] V.Ginzburg, Perverse sheaves on a loop group and Langlands duality, *Preprint alg-geom/9511007*, (1995).

[KS] M.Kashiwara and P.Schapira, Sheaves on manifolds, *Grundlehren der Mathematischen Wissenschaften*, **292**, Springer-Verlag, Berlin, (1994).

[LS] Y.Laszlo and C.Sorger, The line bundles on the moduli of parabolic G-bundles over curves and their sections, *Ann. Sci. cole Norm. Sup.* (4) **30** (1997), no. 4, 499–525.

[Lu] G. Lusztig, Singularities, character formulas, and a q-analogue for weight multiplicities, in " Analyse et topologie sur les espaces singuliers", *Astérisque* **101-102** (1982) 208–229.

[MV] R.MacPherson and K.Vilonen, Elementary construction of perverse sheaves, *Invent. Math.* **84** (1986), no. 2, 403–435.

[MiV1] I.Mirković, K.Vilonen, Characteristic varieties of character sheaves, *Invent. Math.* **93** (1988), no. 2, 405–418.

[MiV2] I.Mirković, K.Vilonen, Perverse sheaves on affine Grassmannians and Langlands duality, *Math. Res. Lett.* **7** (2000), no. 1, 13–24.

[Na] D.Nadler, Perverse Sheaves on Real Loop Grassmannians, *preprint math.AG/0202150*

[PY] G.Prasad, J.K.Yu, Reductive group schemes over discrete valuation rings. preprint 2003.

[PS] A.Pressley, G.Segal, Loop groups, *Oxford Mathematical Monographs*, Oxford University Press, New York, (1986).

[Sa] R.N.Saavedra Catègories Tannakiennes. Lecture Notes in Math. **265**, Springer-Verlag (1972).

[V] K.Vilonen, Perverse sheaves and finite-dimensional algebras, *Trans. Amer. Math. Soc.* **341** (1994), no. 2, 665–676.

Department of mathematics, University of Massachusetts, Amherst, MA 01003, USA
E-mail address: `mirkovic@math.umass.edu`

Department of Mathematics, Northwestern University, Evanston, IL 60208, USA
E-mail address: `vilonen@math.northwestern.edu`