Prerequisites for the Langlands Program

For learning about the Langlands program, knowledge of Lie-group structure theory, algebraic number theory, algebraic geometry, linear algebraic groups, modular forms, and infinite-dimensional representation theory is appropriate. Many people successfully come at the subject having studied at least one of these topics extensively but not necessarily all of them. These notes describe where one can learn a little about each of these topics. Also they include a few deep facts that are not prerequisites but that set the stage for the goals of the program.

It is possible to get a glimpse of the Langlands program, however, with even less background: one needs to understand structure theory of the general linear groups over the real and complex fields; some infinite-dimensional representation theory of these groups, including the nature of the principal series; the fields of p-adic numbers; and something about L-functions in connection with either algebraic number theory or modular forms. References for these particular topics can be extracted from what follows.

Lie-group structure theory

Three relatively recent introductory books about Lie groups, each with its own attitude about the subject, are

Duistermaat, J. J., and J. A. C. Kolk, *Lie Groups*, Springer, 2000,

Rossmann, W., *Lie Groups, an Introduction through Linear Groups*, Oxford, 2002,

Hall, B. C., *Lie Groups, Lie Algebras, and Representations, and Elementary Introduction*, Springer, 2003.

Each of the above books contains a small amount of structure theory. A book at a slightly more advanced level with a great deal more structure theory is

Knapp, A. W., *Lie Groups beyond an Introduction*, Birkhäuser, 1996; second edition, 2002.

The essential structure theory involves complex semisimple Lie algebras, Cartan subalgebras, roots and weights, parabolic subalgebras, reductive groups, Cartan and Iwasawa decompositions, and parabolic subgroups. This material is summarized without proofs in

Knapp, A. W., Structure theory of semisimple Lie groups, Edinburgh proceedings, pp. 1–27, download from

`http://www.math.sunysb.edu/~aknapp/pdf-files/1-27.pdf`.

The expression "Edinburgh proceedings" refers to the volume

Bailey, T. N., and A. W. Knapp (editors), *Representation Theory and Automorphic Forms, Instructional Conference, Edinburgh, 1996*, Proceedings of Symposia in Pure Mathematics, vol. 61, American Mathematical Society, 1997, download table of contents from

`http://www.math.sunysb.edu/~aknapp/books/edinburgh/edinburgh-contents.pdf`.

Algebraic number theory

The relevant algebraic number theory revolves around abelian class field theory and the equality of two kinds of L-functions, one introduced by Artin and the other introduced by Hecke. Abelian class field theory describes concretely the finite Galois extensions with abelian Galois group for a given base field. The eligible base fields are called "global fields" by Weil and are of two kinds—number fields (finite extensions of the rationals) and function fields in one variable over a finite field (finitely generated fields of transcendence degree one over a finite field). A good deal of the development can be done at once for the two kinds of fields; some books take advantage of this fact (not necessarily with better readability on first reading), and others do not.

Often the mathematics in the number-field case is taught in a two-semester course; the first semester treats the basics, and the second semester treats abelian class field theory. There are three traditional books about the number-field case, taking quite different approaches:

Cassels, J. W. S., and A. Fröhlich, *Algebraic Number Theory*, Academic Press, 1967, using an approach emphasizing completeness and cohomology,

Lang, S., *Algebraic Number Theory*, Springer-Verlag, second edition, 1986, using an approach emphasizing a certain amount of complex analysis,

Weil, A., *Basic Number Theory*, Springer-Verlag, 1973, using an approach emphasizing the role of locally compact fields.

Lang's book is limited to number fields, and the other two treat function fields as well. Three techniques of importance in the study of the basics are unique factorization of ideals for Dedekind domains, the theory of discrete valuations, and completion of a global field with respect to a discrete valuation. The completion of a global field with respect to a discrete valuation is called a local field; it is locally compact. The distinct isomorphism classes of completions are called the places of the global field. Some high points of a course on the basics are facts about the discriminant of a number field, the Dirichlet Unit Theorem, and the finiteness of the "class number." The basics occupy Chapters I and II of Cassels–Fröhlich, Chapters I–VI of Lang, and Chapters I–V of Weil.

Adeles and ideles, which are built from all the places of the given global field, enter the theory at some point as convenient tools in preparation for abelian class field theory; the adeles are a kind of product of the ring structures, and the ideles are a kind of product of the multiplicative groups. These tools are introduced at the end of Chapter II of Cassels–Fröhlich, in Chapter VII of Lang, and in the initial development of Weil. The multiplicative group of the global field is a subgroup of each completion, and the multiplicative group therefore embeds diagonally in the group of ideles. The quotient is the all-important "idele class group" of the global field.

For a number field, abelian class field theory may be viewed as built around Artin reciprocity, a sweeping generalization of Gauss's quadratic reciprocity. To any finite abelian Galois extension K of the global field k, the norm map from K to k yields a map from the multiplicative group of K into the idele class group of k, and the image is an open subgroup of finite index. Artin reciprocity allows one to prove that this mapping sets up a bijection between the abelian Galois extensions (up to k isomorphism) and the open subgroups of finite index of the idele class group.

Meanwhile, two kinds of L-functions of one complex variable s are associated to this situation, and they are extremely important to understand for the Langlands program. Each is a product over all places of elementary functions of s. An Artin L-function is associated to each finite Galois extension of k and finite-dimensional complex representation of the Galois group, while a Hecke L-function is associated to each "Grossencharacter," namely each one-dimensional character of the idele class group. The simplest nontrivial examples amount to this: When $k = Q$ and the extension field is $Q(i)$ and the representation of the two-element Galois group of $Q(i)/Q$ is the nontrivial one-dimensional character χ, the Artin L-function is the product over all odd primes p of the expression $\bigl(1 - \chi\bigl(\bigl(\frac{-1}{p}\bigr)\bigr)p^{-s}\bigr)^{-1}$, where $\bigl(\frac{-1}{p}\bigr)$ is the Legendre symbol ± 1; there is also a factor corresponding to the infinite place. Similarly for a suitable Grossencharacter, the Hecke L-function is the product over all odd primes p of $\bigl(1 - \chi((-1)^{(p-1)/2})p^{-s}\bigr)^{-1}$, times a factor corresponding to the infinite place. These two L-functions are equal, factor by factor, because $\bigl(\frac{-1}{p}\bigr) = (-1)^{(p-1)/2}$ by the easiest case of quadratic reciprocity.

In the general case an Artin L-function always encodes certain arithmetic information; it is an instance of what is called a "motivic" L-function. A Hecke L-function encodes certain transformation-group behavior, in that it continues meromorphically to the complex plane and satisfies a functional equation relating its value at s to the value of a companion L-function at $c-s$ for a certain c; it is an instance of what is called an "automorphic" L-function. This automorphic behavior was first proved by Hecke, and Hecke's argument generalized the proof of the functional equation of the Riemann zeta function that uses the Poisson summation formula on the line. Tate in his thesis, which is reproduced in the Cassels–Fröhlich book, showed how to prove the functional equation of the L function of a Grossencharacter by means of the Poisson summation formula for the adele group of the number field.

When an Artin L-function is associated to a one-dimensional representation of an abelian Galois group, Artin reciprocity points to a particular Grossencharacter of interest, and the theorem is that these two L-functions coincide. The motivic L function is therefore exhibited as automorphic. The effort to show that every member of a suitably general class of motivic L functions is automorphic is a theme of the Langlands program. We shall see another instance of this theme in the section "Modular forms."

Abelian class field theory for function fields is related to the geometry of curves, and this geometry is lost to some extent when number fields and function fields are treated together. A book devoted just to the function-field case is

Serre, J.-P., *Algebraic Groups and Class Fields*, Springer-Verlag, 1988.

Comparing this book with the above books by Cassels–Fröhlich and Weil enables one to understand at once both the similarities and the differences between the number-field case and the function-field case. The book

Neukirch, J., *Class Field Theory*, Springer-Verlag, 1986

abstracts abelian class field theory to apply to any base field satisfying certain axioms. The axioms are then verified in the number-field case. The preface says that the theory applies in the function-field case after some modifications, but it urges the reader to study the function-field case in the context of the associated geometry.

Algebraic geometry

The initial need for some algebraic geometry comes from the fact that the above function fields are associated to curves over finite fields. The theory of modular forms to be mentioned below points in addition to curves over any global field and then to arbitrary varieties over such fields. Some basic knowledge of algebraic geometry is therefore appropriate, both for the special theory of curves and for general facts about varieties and maps between varieties.

The above-mentioned book by Serre deals with the function-theoretic properties of curves in Chapter II and with some of the geometric properties in Chapter IV. For both of these chapters it is assumed that the base field is algebraically closed, but later other fields are considered. A recent full-length book about the function-theoretic properties of curves, working over any field, is

Villa Salvador, *Topics in the Theory of Algebraic Function Fields*, Birkhäuser, 2006.

A quite readable and more geometric treatment of curves over algebraically closed fields is

Fulton, W., *Algebraic Curves, An Introduction to Algebraic Geometry*, W. A. Benjamin, Inc., 1969, reprinted, Addison-Wesley, 1974 and 1989.

Fulton's book works extensively with morphisms and puts less emphasis on the function-theoretic aspects than Serre and Villa Salvador. Fulton's book works a little with high-dimensional varieties also.

The book

Hartshorne, R., *Algebraic Geometry*, Springer-Verlag, 1977

begins with a first chapter that quickly develops foundational material about varieties and mappings between them in its Chapter I. Geometric properties of curves are discussed in Section 6 of Chapter I. This subject matter depends a great deal on details in commutative algebra, many of which are proved in the first 35 pages of

Matsumura, H., *Commutative Ring Theory*, Cambridge University Press, 1986; reprinted with corrections, 1989.

Two other books on the elements of algebraic geometry, more leisurely than Hartshorne's, are

Harris, J., *Algebraic Geometry, a First Course*, Springer-Verlag, 1992,

and

Shafarevich, I. R., *Basic Algebraic Geometry*, vol. 1, Springer-Verlag, 1977; second edition, 1994.

Linear algebraic groups

The point of departure for the theory of linear algebraic groups is structure theory for Lie groups, especially for semisimple Lie groups of matrices. The objective is to have a theory in which the real field is replaced by any field, but there are some subtle changes. A good place to get oriented is the obituary of Armand Borel in the *Notices*:

Arthur, J., E. Bombieri, K. Chandrasekharan, F. Hirzebruch, G. Prasad, J.-P. Serre, T. A. Springer, and J. Tits, "Armand Borel (1923–2003)," *Notices of the American Mathematical Society*, vol. 51, no. 5, 2004, 498–524, download from `http://www.ams.org/notices/200405/fea-borel.pdf`.

Each of the eight authors contributed a segment of this article, and the segments by Springer and Tits explain where the theory of algebraic groups comes from and what it wants to do. They assume that one knows in advance a little structure theory for Lie groups. The suggestion is to read the Springer segment first and then the Tits segment.

The Springer and Tits segments of the Borel obituary contain no proofs, and they minimize any mention of the algebraic geometry that underlies the subject. In practice, it is necessary to use algebraic geometry and to adapt its use when the underlying field is not necessarily algebraically closed. Three absolutely standard books on the subject are

Borel, A., *Linear Algebraic Groups*, W. A. Benjamin, 1969; second edition, Springer-Verlag, 1991,

Humphreys, J. E., *Linear Algebraic Groups*, Springer-Verlag, 1975,

Springer, T. A., *Linear Algebraic Groups*, Springer-Verlag, 1981; second edition, 1998.

All three books begin with material on algebraic geometry over an algebraically closed field; Humphreys gives the most detail about this background. Borel and the second edition of Springer give considerable detail about handling groups when the underlying field is not algebraically closed, and Springer's book is the easiest for a first reading (in the view of MathSciNet).

Before these books appeared, the standard expository references were articles in the Boulder proceedings:

Borel, A., and G. D. Mostow, *Algebraic Groups and Discontinuous Subgroups, Boulder, 1965*, Proceedings of Symposia in Pure Mathematics, vol. 9, American Mathematical Society, 1966.

Of particular interest from this volume for learning about algebraic groups are certain articles listed below. These articles show the relative importance of various topics, and they provide examples:

Borel, A., Linear algebraic groups, pp. 3–19, Borel, A., and T. A. Springer, Rationality properties of linear algebraic groups, pp. 26–32,

Tits, J., Classification of algebraic semisimple groups, pp. 33–62,

Bruhat, F., **p**-adic groups, pp. 63–70,

Iwahori, N., Generalized Tits system (Bruhat decomposition) on **p**-adic semisimple groups, pp. 71–83.

Modular forms

A classical modular form of weight k for the group $SL(2,Z)$ is an analytic function f on the upper half plane that is bounded at $i\infty$ and has the property that $f(\gamma z) = (cz+d)^k f(z)$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $SL(2,Z)$ and all z in the upper half plane. A cusp form is a modular form that vanishes at $i\infty$. A beautiful treatment of classical modular and cusp forms appears as Chapter VII of

Serre, J.-P., *A Course in Arithmetic*, Springer-Verlag, 1973.

A cusp form of weight k has a convergent expansion $f(z) = \sum_{n=1}^{\infty} c_n q^n$, where $q = e^{2\pi i z}$, and the L-function of the cusp form is roughly the corresponding series $L(s,f) = \sum_{n=1}^{\infty} c_n n^{-s}$. (Again the definition is not exactly this but includes an additional factor.) This function of s is meromorphic in the whole plane, and the transformation properties of f under $SL(2,Z)$ translate into the fact that $L(s,f)$ satisfies a functional equation relating $L(s,f)$ and $L(k-s,f^\vee)$ for a suitable $f^\vee$. It is thus automorphic in the sense explained in the section "Algebraic number theory." However, further assumptions are needed for $L(s,f)$ to be given by a product expansion over all primes. The conditions are that $c_1 = 1$ and that f be a simultaneous eigenfunction of the commuting family of all "Hecke operators."

What is relevant for current purposes is an extension of this theory in which $SL(2,Z)$ is replaced by its subgroup $\Gamma_0(N)$ of matrices whose lower left entry is divisible by N. The corresponding theory for this subgroup was developed partly by Hecke and was completed by Atkin and Lehner. In addition to the normalization and an eigenfunction property, one needs to impose a condition on the eigenfunction f that it does not come from an eigenfunction of lower N; such an f is called a newform. A detailed exposition can be found can be found in Chapters VIII and IX of

Knapp, A. W., *Elliptic Curves*, Princeton University Press, 1992.

Meanwhile an elliptic curve E over Q can be defined as a nonsingular projective plane curve whose restriction to the affine plane is of the form

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

with coefficients in Q. (In more invariant terms, it is a nonsingular projective curve over Q of genus 1 with a Q rational point.) The coefficients may be taken to be in Z. After a change of variables, the equation may be assumed to have a certain minimality property defined relative to its discriminant Δ. For each prime p, put $a_p = p + 1 - |E_p|$, where E_p is the curve reduced modulo p and $|E_p|$ is the number of points on E_p, including the point at infinity. The L function of E, denoted by $L(s, E)$, is defined as the product over p of a factor equal either to $(1 - a_p p^{-s} + p^{1-2s})^{-1}$ if p does not divide Δ or to $(1 - a_p p^{-s})^{-1}$ if p divides Δ. There is also a factor corresponding to the infinite place. This function and its properties are discussed in Chapter X of the above book.

The Eichler–Shimura theory associates to each normalized newform f of weight 2 for $\Gamma_0(N)$ an elliptic curve E such that $L(s, E) = L(s, f)$. Some of the background for this result appears in Chapter XI of the above book. A complete treatment may be found in

Diamond, F., and J. Shurman, *A First Course in Modular Forms*, Springer-Verlag, 2005.

The Eichler-Shimura result therefore identifies which automorphic L functions associated to cusp forms of $\Gamma_0(N)$ are motivic. The converse, saying that the L function of any elliptic curve over Q is equal to the L function of a normalized newform for some $\Gamma_0(N)$, is known as the Shimura–Taniyama–Weil Conjecture. Wiles proved part of this conjecture in the course of proving Fermat's Last Theorem, and the full conjecture has since been completely proved. The conjecture thus says that the motivic L functions obtained from elliptic curves are always automorphic, and it provides evidence for some of the conjectures of the Langlands program. A relatively short exposition of all these matters appears in

Gelbart, S., Elliptic curves and automorphic representations, *Advances in Math.* 21 (1976), 235–292.

Infinite-dimensional representation theory

The two simple Lie groups of the lowest dimension are $SL(2, R)$ and $SL(2, C)$, and it is sensible to begin to learn representation theory by understanding what happens for these groups. For orientation and for some discussion of these groups, see the first two chapters of

Knapp, A. W., *Representation Theory of Semisimple Groups, An Overview Based on Examples*, Princeton University Press, 1986; reprinted in paperback, 2001.

For more detail, see

Gelfand, I. M., M. I. Graev, and I. I. Pyatetskii-Shapiro, *Representation Theory and Automorphic Forms*, W. B. Saunders, Philadelphia, 1969.

The representation theory of $SL(2,R)$ is discussed in Section 3 of Chapter 1 of the latter book, and the representation theory of $SL(2)$ with entries in any local field is in Sections 3–4 of Chapter 2. For an alternative treatment see

Bump, D., *Automorphic Forms and Representations*, Cambridge University Press, 1997.

In Bump's book, the group $GL(2,R)$ is discussed in part of Chapter 2, and the group $GL(2)$ over a p-adic field is the subject matter of Chapter 4.

Let us turn to groups of higher dimension. The foreword of the Edinburgh proceedings, mentioned in the section "Lie-group structure theory," says that the aim of the conference was "to provide an intensive treatment of representation theory for two purposes: One was to help analysts to make systematic use of Lie groups in work on harmonic analysis, differential equations, and mathematical physics, and the other was to treat for number theorists the representation-theoretic input to Wiles's proof of Fermat's Last Theorem." This representation-theoretic input is via the Langlands program, and these proceedings are a good place to obtain some appropriate background in infinite-dimensional representation theory. The full table of contents can be downloaded from `http://www.math.sunysb.edu/∼aknapp/books/edinburgh/edinburgh-contents.pdf`. The foundational articles on representations of reductive Lie groups, which one can read in order, are

Donley, R. W., Irreducible representations of SL(2,R), pp. 51–59,

Baldoni, W., General representation theory of real reductive Lie groups, pp. 61–72,

Delorme, P., Infinitesimal character and distribution character of representations of reductive Lie groups, pp. 73–81,

Schmid, W., and V. Bolton, Discrete series, pp. 83–113,

Donley, R. W., The Borel–Weil Theorem for $U(n)$, pp. 115–121,

Van den Ban, E. P., Induced representations and the Langlands classification, pp. 123–155,

Mœglin, Representations of GL(n) over the real field, pp. 157–166.

For more detail about many of these topics, one can consult the Knapp book mentioned above.

A place to begin to study the representation theory of groups that are more complicated than $SL(2)$ and $GL(2)$ and are defined over local fields other than R and C is the article

Mœglin, Representations of GL(n) in the nonarchimedean case, Edinburgh proceedings, pp. 303–319.

This latter Mœglin article assumes that the underlying nonarchimedean field has characteristic 0. For groups other than $GL(n)$, once one has some acquaintance with the structure theory of linear algebraic groups, one can consult

Cartier, P., Representations of **p**-adic groups, Corvallis proceedings, Part 1, pp. 111–156.

The expression "Corvallis proceedings" refers to the two volumes

Borel, A., and W. Casselman, *Automorphic Forms, Representations, and L-Functions, Corvallis, 1977*, Proceedings of Symposia in Pure Mathematics, vol. 33, Parts 1 and 2, American Mathematical Society, 1979.

Finally a book-length treatment of the representation theory of reductive groups defined over **p**-adic groups is

Silberger, A. J., *Introduction to Harmonic Analysis on Reductive* **p***-adic Groups*, Princeton University Press, 1979.