

# THE LANGLANDS PROGRAM AND PHYSICS

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ABSTRACT. Notes from the Princeton workshop on the Langlands correspondence and physics

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## 1. INTRODUCTION

The following are notes from the workshop on connections between the Langlands correspondence and Physics that took place at the Institute for Advanced Study at the beginning of March, 2004. Its purpose was to bring together researchers in representation theory and string theory to explore the question of whether it is possible to give a physical perspective on the geometric Langlands correspondence. Certain parts of geometric Langlands make use of tools arising in Conformal Field Theory (CFT), and so provide a point of contact between the two fields. There were several lectures across a period of three days. Mark Goresky gave an introduction to the Langlands correspondence over  $\mathbb{Q}$ , which is the context in which it was first formulated. Edward Frenkel and David Ben-Zvi gave an introduction to the geometric version of the correspondence and its connections with CFT. Finally, Edward Witten gave a lecture on

Langlands duality in physics. These notes aim only to convey a general picture, and many of the statements are not precise.

**Remark:** In the course of preparing these notes, the note-taker benefited greatly from the surveys [F], [G], and [Ge]

## 2. LECTURE 1 - MARK GORESKY THE LANGLANDS CORRESPONDENCE FOR $GL_n(\mathbb{Q})$

Outline

- (1) Approximate statement of conjecture
- (2) Fields and the Galois groups
- (3) Galois representations
- (4) Automorphic forms for  $SL_2$
- (5) Hecke algebra and Hecke operators
- (6) General comments

**2.1. Statement of Conjecture.** The Langlands conjecture for  $GL_n(\mathbb{Q})$  states roughly that there exists a correspondence

$$(2.1) \quad \left\{ \begin{array}{l} \text{"Nice" irreducible } n - \dim \\ \text{representations of } \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{"Nice" cuspidal automorphic} \\ \text{representations of } GL_n(\mathbb{A}_{\mathbb{Q}}) \end{array} \right\}$$

such that

$$\left\{ \begin{array}{l} \text{Eigenvalues of } \text{Fr}_p \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{Eigenvalues of Hecke} \\ \text{operators at } p \end{array} \right\}$$

The rest of this lecture is devoted to explaining the statement and also setting up notation for the other incarnations of the Langlands correspondence.

**2.2. Fields and their Galois Groups.** If  $E \subset F$ , then

**Definition 2.1.**  $\text{Gal}(F/E) = \{ \text{Field automorphisms of } F \text{ fixing } E \text{ pointwise} \}$

*Example 2.1.*  $\mathbb{R} \subset \mathbb{C}$ ,  $\text{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \tau\}$ , where  $\tau$  is complex conjugation

If  $f_1(x_1, \dots, x_n), \dots, f_r(x_1, \dots, x_n)$  is a collection of polynomials in  $E[x_1, \dots, x_n]$ , let

$$X(E) = \{x \in E^n \mid f_i(x) = 0\}$$

be the algebraic variety they define, and

$$X(F) = \{x \in F^n \mid f_i(x) = 0\}$$

the  $F$ -points of  $X$ .  $\text{Gal}(F/E)$  acts on  $X(F)$ , fixing  $X(E)$ .  $\text{Gal}(F/E)$  will also act on various cohomology groups associated to  $X$ . We will now look at some of the fields that arise in the Langlands correspondence.

- **Finite Fields**  $\forall p$  prime, and  $n \geq 1$ , there exists a unique field  $\mathbb{F}(p^n)$  with  $p^n$  elements. We have

$$\mathbb{F}(p^n) \subset \mathbb{F}(p^m) \iff n|m$$

If  $q = p^n$ , then  $\text{Gal}(\mathbb{F}(q^r)/\mathbb{F}(q)) \cong \mathbb{Z}/r\mathbb{Z}$ , and is generated by the Frobenius automorphism  $\text{Fr}_q$ , where

$$\text{Fr}_q : x \mapsto x^q$$

- **Number Fields**

**Definition 2.2.** A *number field* is a finite extension of  $\mathbb{Q}$ .

$\forall f(x) \in \mathbb{Q}[x]$ , solutions to  $f(x) = 0$  lie in a number field

*Example 2.2.* The solutions of the equation  $x^2 = -1$  generate the field

$$\mathbb{Q}(i) = \{a + bi | a, b \in \mathbb{Q}\}$$

Observe that  $\mathbb{Q} = \{\frac{a}{b} | a, b \in \mathbb{Z}\}$ . For a general number field  $E$ , there is an analogue of  $\mathbb{Z}$ , called the ring of integers of  $E$ , and denoted  $\mathcal{O}_E \subset E$ . We have  $E = \{\frac{a}{b} | a, b \in \mathcal{O}_E\}$ . In the above example,  $\mathcal{O}_{\mathbb{Q}(i)} = \mathbb{Z}[i]$ .

We will also need the algebraic closure of  $\mathbb{Q}$ , denoted  $\overline{\mathbb{Q}}$ , where

$$\overline{\mathbb{Q}} = \bigcup_{\text{number fields } E} E$$

- **$p$ -adic fields**

Let

$$\mathbb{Z}_p = \{a_0 + a_1p + a_2p^2 + \dots | 0 \leq a_i \leq p-1\}$$

$\mathbb{Z}_p$  is called the ring of  $p$ -adic integers. Addition and multiplication of these formal series are defined by "carrying". There exists a surjective ring homomorphism

$$\begin{aligned} \mathbb{Z}_p &\mapsto \mathbb{Z}/p\mathbb{Z} \\ \sum a_j p^j &\mapsto a_0 \end{aligned}$$

Let

$$\mathbb{Z}_p^* = \{a_0 + a_1p + a_2p^2 + \dots | a_0 \neq 0\}$$

this is precisely the group of units in  $\mathbb{Z}_p$ . Finally, let

$$\begin{aligned} \mathbb{Q}_p &= \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}_p \right\} \\ &= \{a_m p^m + a_{m+1} p^{m+1} + \dots \mid m > -\infty\} \end{aligned}$$

be the field of  $p$ -adic numbers. Elements can be viewed as formal Laurent series in  $p$ .

**Note:**  $\mathbb{Q}_p$  is a field of characteristic 0.

$\mathbb{Q}_p$  is a completion of  $\mathbb{Q}$  in a certain topology, called the  $p$ -adic topology. In fact, one can show that the only completions of  $\mathbb{Q}$  are  $\mathbb{R}$  and  $\mathbb{Q}_p$  for each prime  $p$ .

**Definition 2.3.** A  $p$ -adic field is a finite extension of  $\mathbb{Q}_p$  for some  $p$ .

- **Function fields** Let  $X$  be an algebraic curve over a field  $F$ . Let  $F(X)$  denote the field of rational functions on  $X$ . In this section,  $F = \mathbb{C}$  or  $\mathbb{F}_q$

*Example 2.3.* If  $F = \mathbb{C}$ , then  $\mathbb{C}(X)$  is the field of meromorphic functions on the Riemann surface  $X$ .

The Galois group of  $F(X)$  is intimately related to the topology of  $X$ . One can show that  $\text{Gal}(\overline{\mathbb{C}(X)}/\mathbb{C}(X))$  is a certain completion of  $\pi_1(X)$ , and  $\text{Gal}(\overline{\mathbb{F}_q(X)}/\mathbb{F}_q(X))$  is the so-called algebraic fundamental group of  $X$ .

**Definition 2.4.** The field of formal Laurent series over  $F$ , denoted  $F((t))$ , is

$$F((t)) = \left\{ \sum_{i > k} a_i t^i \mid a_i \in F, k > -\infty \right\}$$

with addition and multiplication defined in the obvious way.

The subring  $F[[t]]$  is the analogue of the ring of integers  $\mathcal{O}_E$  in the number field case.  $F((t))$  is sometimes called a "local function field" or a "local field". We can view  $F((t))$  as a completion of  $F(X)$  in the following way. Let  $x \in |X|$ , and  $t_x$  a parameter at  $x$ . By expanding a rational function around  $x$  in  $t_x$ , we obtain an embedding

$$F(X) \xrightarrow{e_x, t_x} F((t_x))$$

This yields the "completion at the place  $x$ ". In this sense,  $x$  is analogous to a prime  $p$  in the number field case. As  $x$  varies in  $X$ , we get a family of embeddings of  $F(X)$  into Laurent series. Similarly, as  $p$  varies across all primes, we obtain different embeddings of  $\mathbb{Q}$  into  $\mathbb{Q}_p$ .

**2.3. Galois representations.** We are interested in constructing representations of  $\text{Gal}(\overline{E}/E)$ . One way of constructing such representations is through algebraic geometry. Given an algebraic variety over  $E$ , we can attach to it its étale cohomology groups

$$H_{et}^i(X; \mathbb{Q}_l), \quad l \text{ prime}$$

( $H_{et}^i(X, \overline{\mathbb{Q}}_l) = 0$  if  $i > 2\dim(X)$ ).  $H_{et}^i(X; \overline{\mathbb{Q}}_l)$  is a  $\overline{\mathbb{Q}}_l$ -vector space, and carries a natural action of  $\text{Gal}(\overline{E}/E)$ . If  $E = \mathbb{F}_q$ , then  $\text{Fr}_q \in \text{Gal}(\overline{E}/E)$ . The eigenvalues of  $\text{Fr}_q$  turn out to be very important.

In fact,  $\text{Fr}_p$  acts on representations of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  for every  $p$ . To see why, embed  $\mathbb{Q}$  into  $\mathbb{Q}_p$  and  $\overline{\mathbb{Q}}$  into  $\overline{\mathbb{Q}_p}$ . This yields a homomorphism

$$\phi : \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \mapsto \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$$

There is also a surjective homomorphism

$$\psi : \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \mapsto \text{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$$

It turns out that  $\text{Fr}_p$  can be lifted via  $\psi$  and pushed forward via  $\phi$ . Thus, given an  $n$ -dimensional representation

$$\sigma : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \mapsto GL_n(\overline{\mathbb{Q}}_l)$$

and a prime  $p$ , we can attach to it the  $n$  eigenvalues of  $\text{Fr}_p$

$$(z_1(\sigma_p), \dots, z_n(\sigma_p)), z_i(\sigma_p) \in \bar{\mathbb{Q}}_l)$$

#### 2.4. Automorphic forms and Hecke operators.

**Definition 2.5.** The ring of Adeles of  $\mathbb{Q}$ , denoted  $\mathbb{A}_{\mathbb{Q}}$ , is the subset

$$\{(a_{\infty}, a_2, a_3, a_5, \dots)\} \subset \mathbb{R} \times \prod_p \mathbb{Q}_p$$

such that  $a_p \in \mathbb{Z}_p$  for all but finitely many  $p$ .

**Note:** From now on, we will often abuse notation, and refer to  $\infty$  as a prime.

Let  $GL_n(\mathbb{A}_{\mathbb{Q}})$  denote group of  $n \times n$  invertible matrices with entries in  $\mathbb{A}_{\mathbb{Q}}$ . More explicitly

$$GL_n(\mathbb{A}_{\mathbb{Q}}) = \{(g_{\infty}, g_2, g_3, g_5, \dots)\} \subset GL_n(\mathbb{R}) \times \prod_p GL_n(\mathbb{Q}_p)$$

such that  $g_p \in GL_n(\mathbb{Z}_p)$  for all but finitely many  $p$ .  $\mathbb{Q}$  embeds into  $\mathbb{A}_{\mathbb{Q}}$  diagonally, and induces a diagonal embedding  $GL_n(\mathbb{Q}) \hookrightarrow GL_n(\mathbb{A}_{\mathbb{Q}})$ .

We are interested in constructing representations of  $GL_n(\mathbb{A}_{\mathbb{Q}})$ . When a group acts on a space  $X$ , it automatically acts on  $\mathbb{C}[X]$  - the vector space of complex-valued functions on  $X$ . The homogenous space

$$GL_n(\mathbb{Q}) \backslash GL_n(\mathbb{A}_{\mathbb{Q}})$$

carries a right action of  $GL_n(\mathbb{A}_{\mathbb{Q}})$ , and we will look for representations that can be realized inside  $\mathbb{C}[GL_n(\mathbb{Q}) \backslash GL_n(\mathbb{A}_{\mathbb{Q}})]$ . A "good" representation  $\pi$  is one that decomposes as a tensor product

$$\pi = \pi_{\infty} \otimes \bigotimes_p \pi_p$$

of representations of  $GL_n(\mathbb{R})$  and  $GL_n(\mathbb{Q}_p)$ . We also wish to impose the requirement that each  $\pi_p$  has a one-dimensional subspace  $\mathbb{C} \cdot v_p$  fixed under  $GL_n(\mathbb{Z}_p)$  (the so-called unramified case), and  $\pi_{\infty}$  a one-dimensional subspace fixed by  $SO_n(\mathbb{R})$ . The vector

$$v = v_{\infty} \otimes \bigotimes_p v_p$$

is fixed by  $SO_n(\mathbb{R}) \times \prod_p GL_n(\mathbb{Z}_p)$ . Under the assumption that our representation  $\pi$  is realized inside of  $\mathbb{C}[GL_n(\mathbb{Q}) \backslash GL_n(\mathbb{A}_{\mathbb{Q}})]$ ,  $v$  yields a function on the double coset

$$(2.2) \quad GL_n(\mathbb{Q}) \backslash GL_n(\mathbb{A}_{\mathbb{Q}}) / SO_n(\mathbb{R}) \times \prod_p GL_n(\mathbb{Z}_p)$$

Such a function is called a *cuspidal automorphic function*

2.4.1. *The Hecke algebra.* For each prime  $p$ , let  $\mathcal{H}_p$  denote the algebra of compactly supported  $GL_n(\mathbb{Z}_p)$ -biinvariant functions on  $GL_n(\mathbb{Q}_p)$ . The algebra structure comes from the convolution operation

$$f_1 * f_2 = \int_{GL_n(\mathbb{Q}_p)} f_1(g \cdot g_1) \cdot f_2(g_1^{-1}) d\mu(g_1)$$

$\mathcal{H}_p$  acts (non-trivially) on the one-dimensional space in  $\pi_p$  spanned by  $v_p$  via

$$(2.3) \quad f \cdot v_p = \int_{GL_n(\mathbb{Q}_p)} f(g) g(v_p) d\mu(g)$$

It turns out that  $\mathcal{H}_p$  is abelian and isomorphic to the algebra of symmetric functions on  $(\mathbb{C}^*)^n$ , and so the one-dimensional representation 2.3 is described by the  $n$  eigenvalues

$$(z_1(\pi_p), \dots, z_n(\pi_p)), \quad z_i(\pi_p) \in \mathbb{C}$$

of the generators of  $\mathcal{H}_p$ . One can show that the  $n$ -tuples  $(z_i(\pi_p))$ ,  $1 \leq i \leq n$  completely determine  $\pi$ .

Under the correspondence conjectured in 2.1, if  $\sigma$  is an  $n$ -dimensional Galois representation, and  $\pi^\sigma$  the corresponding automorphic representation, the two sets

$$(z_1(\sigma_p), \dots, z_n(\sigma_p))$$

and

$$(z_1(\pi_p^\sigma), \dots, z_n(\pi_p^\sigma))$$

are supposed to agree for all  $p$ .

**2.5. Connections with classical automorphic forms.** We wish to briefly consider what the theory of automorphic representations (stated in adelic language) has to do with the classical theory of automorphic forms. Let us consider the double quotient 2.2 for  $n = 2$ . Actually, for technical reasons we will look instead at  $SL_n$ , i.e. at

$$(2.4) \quad SL_2(\mathbb{Q}) \backslash SL_2(\mathbb{A}_{\mathbb{Q}}) / SO_2(\mathbb{R}) \times \prod_p SL_2(\mathbb{Z}_p)$$

It is not very hard to see that 2.4 is actually isomorphic to

$$SL_2(\mathbb{Z}) \backslash \mathbb{H}$$

where  $\mathbb{H}$  denotes the upper half-plane. 2.4 therefore has a complex structure, and holomorphic functions (or more generally sections of powers of the canonical line bundle) over 2.4 are modular functions (resp. modular forms) in the classical sense.

### 3. LECTURE 2 - EDWARD FRENKEL FROM FUNCTION FIELDS TO GEOMETRIC LANGLANDS I

In this section, we briefly explain the function field analogue of the Langlands correspondence, and use it to segue into geometric Langlands, by reinterpreting some of the objects that arise in a geometric manner. Let  $X$  be a smooth proper curve over  $\mathbb{F}_q$  where  $q = p^n$  for some prime  $p$ , with function field  $F = \mathbb{F}_q(X)$ , and let

$$G_F = \text{Gal}(\overline{F}/F)$$

denote its Galois group - it can be viewed as the group of deck transformations of the maximal ramified cover of  $X$ , and as such is a certain analogue of the fundamental group in geometry. Let  $|X|$  denote the set of closed points of  $X$ , or equivalently, the set of  $\overline{\mathbb{F}_q}$ -points of  $X$ . If  $x \in |X|$ , the residue field  $\mathbb{F}_{q_x}$  at  $x$  is a finite extension of  $\mathbb{F}_q$ . Denote by  $\mathcal{O}_x$  the completion of the local ring at  $x$ , and by  $F_x$  the field of fractions of  $\mathcal{O}_x$ . If we choose a local parameter  $t_x$  at  $x$ , then  $\mathcal{O}_x = \mathbb{F}_{q_x}[[t_x]]$ , and  $F_x = \mathbb{F}_{q_x}((t_x))$ . Let

$$\mathbb{A}_X = \{(f_x) \in \prod_{x \in |X|} \mathbb{F}_x \mid f_x \in \mathcal{O}_x \text{ for all but finitely many } x\}$$

be the ring of adèles of  $F$ , and

$$\mathbb{O}_X = \{(f_x) \in \mathbb{A}_X \mid f_x \in \mathcal{O}_x \forall x\}$$

the integral adèles.  $F$  embeds diagonally into  $\mathbb{A}_X$  by expanding rational functions around points  $x \in |X|$

Just like the case of number fields, the Langlands correspondence for function fields will have a Galois side and an automorphic representation side. On the Galois side, we are interested in  $n$ -dimensional irreducible  $l$ -adic representations

$$G_F \mapsto GL_n(\overline{\mathbb{Q}_l})$$

(the reason why we use  $\overline{\mathbb{Q}_l}$  rather than  $\mathbb{C}$  is that the class of complex representations is unnecessarily restrictive). On the automorphic side, we are interested in representations of  $GL_n(\mathbb{A}_X)$  that can be realized inside the space of compactly supported functions on the quotient

$$(3.1) \quad GL_n(F) \backslash GL_n(\mathbb{A}_X)$$

More precisely, we wish to consider are the cuspidal automorphic representations, which are those representations realized inside the subspace  $\mathcal{C}_{cusp}$  of functions on 3.1 satisfying the condition that for each choice of  $n_1, n_2$  satisfying  $n_1 + n_2 = n$ ,

$$(3.2) \quad \int_{U_{n_1, n_2}(F) \backslash U_{n_1, n_2}(\mathbb{A}_X)} \phi(ug) du = 0, \quad \forall g \in GL_n(\mathbb{A})$$

where  $U_{n_1, n_2}$  denotes the unipotent radical of the standard parabolic subgroup of  $GL_n$  corresponding to the partition  $n = n_1 + n_2$ . Under the right action of  $GL_n(\mathbb{A})$ ,  $\mathcal{C}_{cusp}$  decomposes into a direct sum of irreducible representations, each of which occurs with multiplicity one. These are the desired cuspidal automorphic reps.

Suppose now that  $\pi$  is an irreducible cuspidal automorphic representation of  $GL_n(\mathbb{A})$ . We wish to decompose it with respect to each place  $x \in |X|$ . One can show that it decomposes into the restricted tensor product

$$(3.3) \quad \pi = \bigotimes_{x \in |X|} ' \pi_x$$

where  $\pi_x$  is a representation of  $GL_n(F_x)$ , and for  $x \in |X| \setminus S$ ,  $S$  a finite set,  $\pi_x$  contains a vector  $v_x$  invariant with respect to the subgroup  $GL_n(\mathcal{O}_x)$ .  $S$  is the set of places where  $\pi$  is said to be *ramified*. One can show that  $v_x$  is unique up to scalar multiplication.

The right-hand side of 3.3 is defined as the span of vectors of the form

$$(3.4) \quad w = \bigotimes_{x \in |X|} w_x$$

where  $w_x = v_x$  for all but finitely many  $x \in |X| \setminus S$ .

It turns out that the representation  $\pi$  can be uniquely described in terms of the representation theory of a much simpler object than  $GL_n(\mathbb{A})$ , called the spherical Hecke algebra, and denoted  $\mathcal{H}(GL_n(\mathbb{A}), GL_n(\mathcal{O}))$  or simply  $\mathcal{H}$ . We first describe the local version  $\mathcal{H}_x = \mathcal{H}(GL_n(F_x), GL_n(\mathcal{O}_x))$  of  $\mathcal{H}$  - it is by definition the algebra of compactly supported functions on  $GL_n(F_x)$  that are bi-invariant (i.e. left and right - invariant) with respect to the subgroup  $GL_n(\mathcal{O}_x)$ . The algebra structure is given by the convolution product

$$f_1 * f_2(g) = \int_{GL_n(F_x)} f_1(g \cdot g_1) \cdot f_2(g_1^{-1}) d\mu(g_1)$$

where  $\mu$  denotes Haar measure on  $GL_n(F_x)$  normalized so that  $\mu(GL_n(\mathcal{O}_x)) = 1$ . If  $x \in |X| \setminus S$ , then  $\mathcal{H}_x$  acts on the one-dimensional subspace  $\pi_x^{GL_n(\mathcal{O}_x)}$  via

$$(3.5) \quad f \cdot v_x = \int_{GL_n(F_x)} f(g) \cdot g(v_x) d\mu(g)$$

(for an argument why this action preserves the line spanned by  $v_x$  see [G]). It is known that  $\mathcal{H}_x$  is commutative, and isomorphic to

$$\overline{\mathbb{Q}_l}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]^{S_n}$$

Note that this is the algebra of Weyl-group invariant functions on the maximal torus of  $GL_n(\overline{\mathbb{Q}_l})$ , whose spectrum corresponds to the set of semisimple conjugacy classes in  $GL_n$  (this description generalizes to other groups as well).  $\mathcal{H}_x$  therefore acts on the one-dimensional space spanned by  $v_x$  through a character, described by its eigenvalues

$$(z_1(\pi_x), \dots, z_n(\pi_x))$$

The global version of the Hecke algebra is defined as

$$\mathcal{H} = \mathcal{H}(GL_n(\mathbb{A}), GL_n(\mathcal{O})) \stackrel{def}{=} \bigotimes'_{x \in |X|} \mathcal{H}_x$$

where the  $'$  denotes the subset spanned by tensors  $\otimes f_x$ , where  $f_x = 1$  for all but finitely many  $x$ .

For each place  $x \in |X|$ , the Galois group  $Gal(\overline{\mathbb{F}_{q_x}}/\mathbb{F}_{q_x})$  contains the Frobenius element  $Fr_x$ , defined to be the inverse of the automorphism  $y \mapsto y^{q_x}$ . Provided that the Galois representation

$$\sigma : G_F \mapsto GL_n(\overline{\mathbb{Q}_l})$$

is unramified at  $x$ ,  $Fr_x$  gives rise to a well-defined semisimple conjugacy class in  $GL_n(\overline{\mathbb{Q}_l})$ , described by its eigenvalues

$$(z_1(\sigma_x), \dots, z_n(\sigma_x))$$

We can now state the Langlands correspondence for  $GL_n$  over function fields, which was established by Drinfeld in the case  $n = 2$  and by Lafforgue for general  $n$ .

**Theorem 3.1.** *There exists a bijection*

$$\left\{ \begin{array}{l} \text{Irreducible representations of} \\ G_F \mapsto GL_n(\overline{\mathbb{Q}}_l) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{Irreducible cuspidal automorphic} \\ \text{representations of } GL_n(\mathbb{A}) \end{array} \right\}$$

such that for corresponding  $\sigma$  and  $\pi$ , for every place  $x \in |X| \setminus S$

$$(z_1(\sigma_x), \dots, z_n(\sigma_x)) \leftrightarrow (z_1(\pi_x), \dots, z_n(\pi_x))$$

Suppose for a moment that  $\pi$  is unramified, i.e. that  $S$  is empty.  $\pi$  then contains a one-dimensional space invariant with respect to  $GL_n(\mathbb{O})$ , which in the decomposition 3.3 is spanned by the vector

$$v = \bigotimes_{x \in |X|} v_x$$

$v$  is therefore a cuspidal function that descends to the double quotient

$$(3.6) \quad GL_n(F) \backslash GL_n(\mathbb{A}) / GL_n(\mathbb{O})$$

Such a function is called emph cuspidal automorphic. One can pass between cuspidal automorphic functions and unramified representations by on one hand taking  $GL_n(\mathbb{O})$ -invariants, and on the other hand the subspace generated by  $v$  in  $\mathcal{C}_{cusp}$  under the  $GL_n(\mathbb{A})$  action. The geometric interpretation of the space 3.6 is the point of departure for the geometric Langlands correspondence. We have the following lemma, whose statement and proof are taken from [F]:

**Lemma 3.1.** *There is a bijection between 3.6 and the set of isomorphism classes of rank  $n$  vector bundles on  $X$ .*

*Proof.* Any rank  $n$  bundle on  $X$  may be trivialized on the formal disc  $D_x = \text{Spec } \mathcal{O}_x$  around each point  $x \in |X|$  and over  $X - S$ , where  $S$  is a sufficiently large subset of  $|X|$ , hence over the generic point  $X_{\text{gen}} = \text{Spec } F$  of  $X$ .

Let  $B$  be the set of isomorphism classes of the data  $(\mathcal{M}, \varphi_{\text{gen}}, (\varphi_x))$ , where  $\mathcal{M}$  is a rank  $n$  bundle on  $X$ , and  $\varphi_{\text{gen}}$  and  $\varphi_x, x \in |X|$ , are the trivializations of  $\mathcal{M}$  over the generic point of  $X$  and the formal discs  $D_x$ , respectively, i.e., isomorphisms  $\varphi_{\text{gen}} : F^n \xrightarrow{\sim} \Gamma(X_{\text{gen}}, \mathcal{M})$ ,  $\varphi_x : \mathcal{O}_x^n \xrightarrow{\sim} \Gamma(D_x, \mathcal{M})$ . The restrictions of  $\varphi_{\text{gen}}$  and  $\varphi_x$  to the punctured disc  $D_x^\times = \text{Spec } \mathcal{K}_x$  then give us two different trivializations of  $\mathcal{M}|_{D_x^\times}$ , which we denote by the same symbols. Let  $g_x = \varphi_{\text{gen}}^{-1} \circ \varphi_x$  be the corresponding transition function, which is an element of  $GL_n(\mathcal{K}_x)$ .

Consider the map  $b : B \rightarrow GL_n(\mathbb{A})$  sending  $(\mathcal{M}, \varphi_{\text{gen}}, (\varphi_x))$  to  $(g_x)_{x \in |X|} \in GL_n(\mathbb{A})$ . It is easy to see that this map is a bijection. Therefore there is bijection between the set of isomorphism classes of rank  $n$  bundles on  $X$  and the quotient of  $GL_n(\mathbb{A})$  by the equivalence relations corresponding to changes of the trivializations  $\varphi_{\text{gen}}$  and  $\varphi_x, x \in |X|$ . These equivalence relations amount to the left action of  $GL_n(F)$  and the right action of  $GL_n(\mathbb{O})$  on  $GL_n(\mathbb{A})$ , respectively. Hence we obtain the statement of the lemma. □

After interpreting the double quotient 3.6 as the moduli of rank  $n$  vector bundles on  $Xt$ , the next step towards geometric Langlands is to replace functions by some

suitable geometric object. Note that the moduli of vector bundles is an object that makes sense not only over  $\mathbb{F}_q$ , but also  $\mathbb{C}$ , and so one can formulate a version of the correspondence over the complex numbers as well. There are two candidates for a geometric replacement of the notion of cuspidal automorphic function - complexes of  $l$ -adic sheaves, and  $D$ -modules. The former is better adapted for the setting of curves  $\mathbb{F}_q$ , and the latter for curves over  $\mathbb{C}$ .

Let  $E$  be a complex of  $l$ -adic sheaves on a scheme  $Y$  over  $\mathbb{F}_q$ . Denote by  $Y(\mathbb{F}_q)$  the set of its  $\mathbb{F}_q$ -points, and by  $\text{Fun}(Y)$  the space of  $\overline{\mathbb{Q}_l}$ -valued functions on  $Y(\mathbb{F}_q)$ . The Galois group  $\text{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$  acts on  $E$ , and we can assign to  $E$  an element of  $\phi_E \in \text{Fun}(Y)$  defined by

$$(3.7) \quad \phi_E(y) = \sum_i (-1)^i \text{Tr}(\text{Fr}_y, E_y^i), \quad y \in Y(\mathbb{F}_q)$$

The assignment 3.7 is called *Grothendieck's function-sheaf dictionary*. One hopes that interesting functions on 3.6, such as our cuspidal automorphic function  $v$ , come from some complex  $E$  via the correspondence 3.7. In fact, the assignment is well-defined on  $D^b(Y)$  - the bounded derived category of  $l$ -adic complexes on  $Y$  with constructible cohomologies. The category  $D^b(Y)$  admits various natural operations, such as pull-backs, push-forwards, tensor products, etc. One can show that

- If  $E_1, E_2$  are two objects of  $D^b(Y)$ , then  $\phi_{E_1 \otimes E_2} = \phi_{E_1} \cdot \phi_{E_2}$  (where the RHS denotes pointwise multiplication).
- If  $f : Y \mapsto Z$  is a morphism, and  $F \in D^b(Z)$  then  $\phi_{f^* F} = f^* \phi_F$
- $\phi_{f_*(E_1)}$  corresponds to integration of  $\phi_{E_1}$  along the fibers.

The other "geometrization" of a function is the notion of a  $D$ -module. The idea is basically to replace a function by the system of linear differential equations that it satisfies. For a variety  $Y$ , we denote by  $D_Y$  its sheaf of algebraic differential operators. Locally, in coordinates  $y_1, \dots, y_k$ , such a differential operator can be expressed as a sum of the form

$$\sum_{\alpha} a^{\alpha}(y) \partial^{\alpha} / \partial y^{\alpha}$$

where  $a^{\alpha}(y)$  are algebraic functions of the  $y_i$ 's, and  $\alpha$  denotes multi-indices. A (left)  $D_Y$ -module is a sheaf of left modules over  $D_Y$  (one defines a right  $D_Y$ -module similarly). To see how the notion of a differential equation can be formulated in terms of  $D$ -modules, let  $P_1, \dots, P_m$  be differential operators, and let  $\mathcal{J}$  be the left ideal they generate. A homomorphism

$$D_Y / \mathcal{J} \mapsto \mathcal{F}$$

to a  $D_Y$ -module  $\mathcal{F}$  (e.g. functions, distributions, etc.) is the same as a solution in  $\mathcal{F}$  of the system

$$P_i \cdot \xi = 0 \quad \forall i$$

The next step is to geometrize the action of the Hecke algebra. Let  $\mathcal{H}ecke_{x,i}$  denote the set of isomorphism classes of triples  $(\mathcal{M}, \mathcal{M}', \beta)$ , where  $\mathcal{M}, \mathcal{M}'$  are rank  $n$  vector bundles on  $X$ , and  $\beta$  is an embedding of the coherent sheaves of sections of  $\mathcal{M}$  into  $\mathcal{M}'$ , such that the quotient  $\mathcal{M}'/\mathcal{M}$  has length  $i$  and is scheme-theoretically supported at  $x$ . We can put the  $\mathcal{H}ecke_{x,i}$  together for different  $x \in |X|$  to obtain  $\mathcal{H}ecke_i$ .  $\mathcal{H}ecke_i$

comes with maps  $h^\leftarrow, h^\rightarrow$  to  $\text{Bun}_n$ , where  $h^\leftarrow(\mathcal{M}, \mathcal{M}', \beta) = \mathcal{M}$ ,  $h^\rightarrow(\mathcal{M}, \mathcal{M}', \beta) = \mathcal{M}'$ , and a map  $\text{supp}$  to  $|X|$  sending the triple to the point  $x$  where the modification happens. For each  $1 \leq i \leq n$ , we thus have the following picture:

$$\begin{array}{ccc} & \mathcal{H}ecke_i & \\ h^\leftarrow \swarrow & & \searrow \text{supp} \times h^\rightarrow \\ \text{Bun}_n & & X \times \text{Bun}_n \end{array}$$

It is easy to see that  $(h^\leftarrow)^{-1} \cap \text{supp}^{-1}(x)$  in  $\mathcal{H}ecke_i$  is the set of  $i$ -dimensional subspaces in  $\mathcal{M}_x$  (just assign to the triple the fiber of  $\ker(\beta)$  at  $x$ ), and  $(h^\rightarrow)^{-1} \cap \text{supp}^{-1}(x)$  is the set of  $n - i$  dimensional subspaces of  $\mathcal{M}'_x$  (assign to the triple the image of  $\mathcal{M}_x$  inside  $\mathcal{M}'_x$ ). Let us restrict our attention for a moment to  $\mathcal{H}ecke_{x,i}$ , and consider the operation on functions on  $\text{Bun}_n$  given by

$$H_{i,x} : f \mapsto h_*^\rightarrow (h^\leftarrow)^*(f), \quad f \in \mathbb{C}[\text{Bun}_n]$$

One can check that the action of  $H_{i,x}$  agrees with the action 3.5 of  $\chi_i$ , where  $\chi_i$  is the characteristic function of the double coset

$$GL_n(\mathcal{O}_x) \text{Diag}(t_x, \dots, t_x, 1, \dots, 1) GL_n(\mathcal{O}_x)$$

where the uniformizer  $t_x$  occurs  $i$  times.

The cuspidality condition 3.2 can also be phrased in terms of correspondences. Suppose that  $P_{n_1, n_2} \subset GL_n$  is the parabolic subgroup corresponding to the partition  $n = n_1 + n_2$ , and let  $\text{Bun}_{n_1, n_2}$  denote the moduli space of  $P_{n_1, n_2}$ -bundles. Note that a  $P_{n_1, n_2}$ -bundle is the same as an exact sequence of vector bundles

$$(3.8) \quad 0 \mapsto \mathcal{M}_1 \mapsto \mathcal{M} \mapsto \mathcal{M}_2 \mapsto 0$$

where  $\mathcal{M}_1, \mathcal{M}, \mathcal{M}_2$  are of ranks  $n_1, n$ , and  $n_2$  respectively. We have the picture

$$\begin{array}{ccc} & \text{Bun}_{n_1, n_2} & \\ p \swarrow & & \searrow q \\ \text{Bun}_n & & \text{Bun}_{n_1} \times \text{Bun}_{n_2} \end{array}$$

where  $p$  sends the sequence 3.8 to  $\mathcal{M}$  and  $q$  to the pair  $(\mathcal{M}_1, \mathcal{M}_2)$ . One can check that a function on  $\text{Bun}_n$  is cuspidal iff

$$q_*(p^*(f)) = 0$$

for all partitions  $n_1 + n_2 = n$ ,  $n_i > 0$ .

The operations  $(h^\leftarrow)^*, (h^\rightarrow)_*$  make sense in the derived category of  $l$ -adic complexes or  $D$ -modules. In trying to come up with the correct analogue of a cuspidal automorphic function in  $D^b(\text{Bun}_n)$  or  $D_Y(\text{mod})$ , we should keep in mind that (at least in the unramified case) such a function is an eigenfunction for the entire Hecke algebra. We are therefore looking for an "eigen-sheaf" for the Hecke operators  $H_{x,i}$ ,  $\forall x \in X, 1 \leq i \leq n$ . We can instead work with  $\mathcal{H}ecke_i$  and put all of the Hecke operators for different  $x$ 's together into operators  $H_i$ .

**Definition 3.1.** Let  $E$  be a local system on the curve  $X$ . We say that a complex  $\text{Aut}_E \in D^b(\text{Bun}_n)$  (resp. a  $D_{\text{Bun}_n}$ -module) is a Hecke eigensheaf (resp. eigen  $D$ -module) with eigenvalue  $E$ , if

$$(3.9) \quad H_i(\text{Aut}_E) = \bigwedge^i E \boxtimes \text{Aut}_E, \quad 1 \leq i \leq n$$

In the case where the base field is  $\mathbb{F}_q$ , we can apply the function-sheaf correspondence to the equation 3.9, and observe that the values of the function  $\phi_E$  corresponding to  $\text{Aut}_E$  are symmetric functions of the Frobenius eigenvalues on the stalks of  $E$ , which precisely implies that  $\phi_E$  is automorphic in the classical sense. We can now state the geometric Langlands correspondence (which makes sense over both  $\mathbb{F}_q$  and  $\mathbb{C}$ ) for  $GL_n$ , which has been proven by Frenkel, Gaitsgory, and Vilonen.

**Theorem 3.2.** *For each rank  $n$  local system on  $X$ , there exists a cuspidal Hecke eigensheaf  $\text{Aut}_E$  on  $\text{Bun}_n$  with eigenvalue  $E$ .*

**3.1. Geometric Langlands and CFT.** Let us now restrict our attention to the case of  $F = \mathbb{C}$ . We are interested in producing Hecke  $D$ -modules  $\text{Aut}_E$  on  $\text{Bun}_n$ . Such an object corresponds to a system of differential equations on  $\text{Bun}_n$ . One way of producing systems of differential equations on moduli spaces associated to a curve  $X/\mathbb{C}$  (such as  $\text{Bun}_n$  or  $\mathcal{M}_{g,n}$ , the moduli of curves) is through Conformal Field Theory (CFT). The idea is that a CFT gives rise to objects known as chiral correlation functions, which are horizontal sections of certain vector bundles with projectively flat connection over our moduli space. The connection is called the Knizhnik-Zamolodchikov (KZ) connection, and the system of differential equations is equivalent to the horizontality condition.

The mathematical object that encodes the algebraic content of a CFT is known as a vertex algebra. Let  $G$  be a complex algebraic group, and  $\text{Bun}_G$  denote the moduli stack of  $G$ -bundles on  $X$ . Given a vertex algebra  $V$  possessing symmetries of an algebraic group  $G$ , there exists a *localization functor*

$$\Delta_V : (V\text{-mod}) \mapsto (D_{\text{Bun}_G}\text{-mod})$$

that sends a  $V$ -representation  $M$  to the  $D$ -module  $\Delta_V(M)$  on  $\text{Bun}_G$  (actually, instead of  $D$ -modules we may obtain twisted  $D$ -modules in general).

Conjecturally, the construction of Hecke  $D$ -modules on  $\text{Bun}_G$  should be accomplished by applying the localization functor to the *chiral Hecke algebra*, which is a vertex algebra introduced by Beilinson and Drinfeld.

#### 4. LECTURE 3 - DAVID BEN-ZVI THE GEOMETRIC LANGLANDS CORRESPONDENCE FOR $GL_1$

In this lecture we will look at the simplest case of geometric Langlands, namely that of  $GL_1$ . This case is often referred to as geometric class field theory, in analogy with the number field case. We are thus looking to establish the following correspondence:

$$\{\text{Rank 1 local systems on } X\} \leftrightarrow \{\text{Hecke eigensheaves on } \text{Bun}_1\}$$

$$E \leftrightarrow \text{Aut}_E$$

$\text{Bun}_1 = \text{Pic}(X)$  is isomorphic to  $\text{Jac}(X) \times \mathbb{Z}$ , where

$$(4.1) \quad \text{Jac}(X) = \text{Pic}_0(X) = H^1(X, \mathcal{O})/H^1(X, \mathbb{Z})$$

Our strategy is to construct  $\text{Aut}_E$  over  $\text{Jac}(X)$ , and then to extend it to the other components of  $\text{Bun}_1$ . A rank 1 local system  $E$  is equivalent to a one-dimensional representation  $L_E$  of the fundamental group of  $X$

$$L_E : \pi_1(X) \mapsto \text{GL}_1(\mathbb{C}) = \mathbb{C}^*$$

Because the target is abelian,  $L_E$  necessarily factors through the abelianization  $H_1(X, \mathbb{Z})$  of  $\pi_1(X)$ . Now,  $H_1(X, \mathbb{Z}) \cong H^1(X, \mathbb{Z})$  by Poincaré duality, and from 4.1 we see that  $H^1(X, \mathbb{Z}) = \pi_1(\text{Jac}(X))$ . Thus  $L_E$  induces a character of  $\pi_1(\text{Jac}(X))$ , and therefore a rank one local system on  $\text{Jac}(X)$  which we denote  $\text{Aut}_E$ . We now have to extend  $\text{Aut}_E$  to  $\text{Bun}_1$  in such a way that it becomes a Hecke eigensheaf. We have the Hecke correspondence

$$\begin{array}{ccc} & \text{Hecke}_1 & \\ h^\leftarrow \swarrow & & \text{supp} \times h^\rightarrow \searrow \\ \text{Bun}_1 & & X \times \text{Bun}_1 \end{array}$$

$\text{Hecke}_1 = \{(x, \mathcal{L}, \mathcal{L}')\}$  where  $\mathcal{L}, \mathcal{L}' \in \text{Bun}_1$ , and the sheaves of sections (which we denote by the same letters) fit into an exact sequence:

$$(4.2) \quad 0 \mapsto \mathcal{L}' \mapsto \mathcal{L} \mapsto \mathcal{O}_x \mapsto 0$$

with  $\mathcal{O}_x$  denoting the skyscraper sheaf at  $x$ . The maps in the diagram are defined as follows:  $h^\leftarrow(x, \mathcal{L}, \mathcal{L}') = \mathcal{L}$ ,  $\text{supp}(x, \mathcal{L}, \mathcal{L}') = x$ ,  $h^\rightarrow(x, \mathcal{L}, \mathcal{L}') = \mathcal{L}'$ . The condition 4.2 uniquely determines  $\mathcal{L}$  and  $\mathcal{L}'$ , and one can show that  $\mathcal{L} = \mathcal{L}'(x)$ . It follows that

$$(4.3) \quad \text{Hecke}_1 \cong X \times \text{Bun}_1$$

and that  $\text{supp} \times h^\rightarrow$  is an isomorphism. Under the isomorphism 4.3, we can identify  $h^\leftarrow$  with the map sending  $(x, \mathcal{L})$  to  $\mathcal{L}(x)$ . The Hecke eigensheaf condition is simply

$$(h^\leftarrow)^*(\text{Aut}_E) = E \boxtimes \text{Aut}_E$$

and uniquely determines the extension of  $\text{Aut}_E$  from  $\text{Jac}$  to  $\text{Bun}_1$ .

## 5. LECTURE 4 - DAVID BEN-ZVI

### GEOMETRIC LANGLANDS AS A QUANTIZATION OF THE HITCHIN SYSTEM

The purpose of this lecture is to explain how a classical integrable system, called the Hitchin system, can be viewed as a quasi-classical limit of the geometric Langlands correspondence for a subset of local systems, called opers. We begin with some general remarks about quantization. Given a filtered algebra  $A$  (usually, but not always non-commutative), we say that its associated graded  $gr(A)$ , a commutative Poisson algebra, is the quasi-classical limit of  $A$ . Conversely, we will say that  $A$  is a quantization of  $gr(A)$ .

*Example 5.1.* Take  $A$  to be the Weyl algebra  $W_{N,h}$  on  $N$  generators - this is the algebra generated by

$$x_1, \dots, x_N, \partial_{x_1}, \dots, \partial_{x_n}, 1$$

subject to the usual commutation relations  $[\partial_{x_i}, x_j] = \delta_{i,j}h$ . For  $h = 1$ , this is the algebra of differential operators on  $N$ -dimensional affine space  $\mathbb{A}_N$ . It is filtered by the order of the differential operator, and

$$gr(A) \cong \mathbb{C}[x_1, \dots, x_N, y_1, \dots, y_N] = \mathbb{C}[T^*\mathbb{A}_N]$$

together with the Poisson structure induced from the standard symplectic structure on the cotangent bundle to a manifold.

*Example 5.2.* In this example,  $A$  will be commutative. Let  $V$  be a vector space, and  $W$  an affine space for  $V$ , i.e.  $W$  looks like  $V$  but has no distinguished origin (one could also say that  $W$  is a  $V$ -torsor).  $\mathbb{C}[V]$ , the algebra of functions on  $V$ , has a canonical grading by assigning to a function  $f$  the order of vanishing at the origin. Here,  $\mathbb{C}[V] = gr(A)$ , where  $A = \mathbb{C}[W]$ , and  $A$  carries the obvious filtration.

**5.1. The Hitchin system.** We fix  $G$ , a simple complex algebraic group or  $G = GL_n$ . The cotangent bundle of  $\text{Bun}_G$  has the following simple description:

$$T^* \text{Bun}_G = \{(\mathcal{P}, \eta) \mid \mathcal{P} \in \text{Bun}_G, \eta \in H^0(X, \text{ad}(\mathfrak{g}) \otimes \Omega_X)\}$$

where

$$\text{ad}(\mathfrak{g}) = \mathcal{P} \times_G \mathfrak{g}$$

*Example 5.3.* For  $G = GL_n$ , identifying  $\text{Bun}_{GL_n}$  with  $\text{Bun}_n$ ,

$$T^* \text{Bun}_n = \{(\mathcal{E}, \eta) \mid \mathcal{E} \in \text{Bun}_n, \eta \in H^0(X, \text{End}(\mathcal{E}) \otimes \Omega_X)\}$$

There is a map

$$\mu : T^* \text{Bun}_G \mapsto \bigoplus_{d_i} H^0(\Omega^{\otimes(d_i+1)})$$

called the Hitchin moment map, where the  $d_i$  are the exponents of  $\mathfrak{g}$ . The space on the right is called the Hitchin space, and denoted  $\text{Hitch}_G$ .

*Example 5.4.* In example 5.3, the exponents are  $d_i = i - 1$ , so the Hitchin space is just

$$\bigoplus_{i=1}^n H^0(\Omega^{\otimes i})$$

The generic fibers of  $\mu$  are abelian varieties, which implies that the only functions on  $T^* \text{Bun}_G$  are those that are pulled back from  $\text{Hitch}_G$ . Furthermore, the pullbacks Poisson commute with respect to the canonical symplectic structure on  $T^* \text{Bun}_G$ . We summarize:

**Theorem 5.1.** *We have*

$$(5.1) \quad \mathbb{C}[\text{Hitch}_G(X)] \stackrel{\mu^*}{\cong} \mathbb{C}[T^* \text{Bun}_G]$$

*and the images Poisson commute.*

We now want to quantize the isomorphism 5.1 along the lines of the two examples 5.1 and 5.2. The idea is to:

- replace the right hand side of 5.1 by an algebra of differential operators on  $\text{Bun}_G$
- replace left hand side by an algebra of functions on an affine space for  $\text{Hitch}_G$ , called the space of  ${}^L G$ -opers, and denoted  $\text{Op}_{{}^L G}$

Unfortunately,  $\text{Bun}_G$  does not carry any non-trivial global differential operators, and instead we consider twisted differential operators  $D_k$  acting on the  $k$ -th power of the determinant line bundle  $\mathcal{L}$  over  $\text{Bun}_G$ . We have the following result, which can be viewed as the quantization of the isomorphism 5.1

$$(5.2) \quad \Gamma(\text{Bun}_G, D_k) = \begin{cases} \mathbb{C}, & k \neq -\check{h} \\ \mathbb{C}[\text{Op}_{{}^L G}], & k = -\check{h} \end{cases}$$

where  $\text{Op}_{{}^L G}$  denotes the space of  ${}^L G$  *opers* introduced in E. Frenkel's lectures. We have the following commutative diagram

$$\begin{array}{ccc} \Gamma(\text{Bun}_G, D_{-\check{h}}) & \longrightarrow & \mathbb{C}[\text{Op}_{{}^L G}] \\ \text{symp} \downarrow & & \downarrow \text{gr} \\ \Gamma(T^* \text{Bun}_G, \mathcal{O}_{T^* \text{Bun}_G}) & \longrightarrow & \mathbb{C}[\text{Hitch}_G(X)] \end{array}$$

where *symp* denotes the map assigning to a (twisted) differential operator its symbol. Under *symp*, symbols of global differential operators on  $\text{Bun}_G$  map to the Hitchin hamiltonians. We now examine how this picture relates to the geometric Langlands correspondence. As explained in [F2], an oper  $\rho \in \text{Op}_{{}^L G}$  is a special kind of local system, and under the isomorphism 5.2 yields a functional on  $D_{-\check{h}}$ . The  $D_{-\check{h}}$ -module

$$M_\rho = D_{-\check{h}} / D_{-\check{h}} \langle D_i - \rho(D_i) \rangle$$

is the desired Hecke eigensheaf corresponding to  $\rho$  (actually, we should twist by  $\omega^{-1/2}$ , where  $\omega$  is the canonical line bundle on  $\text{Bun}_G$ ). Over an open subset of  $\text{Bun}_G$ ,  $M_\rho$  is the sheaf of sections of a vector bundle with projectively flat connection. The "classical limit" of  $M_\rho$  is the  $\mathcal{O}_{T^* \text{Bun}_G}$ -module

$$\mathcal{O}_\chi = \mathcal{O}_{T^* \text{Bun}_G} / \langle \xi_i - \chi(\xi_i) \rangle$$

where  $\chi$  is the corresponding element of  $\text{Hitch}_G$ . The latter is isomorphic to the sheaf of functions on the fiber  $\mu^{-1}(\chi)$ .

So far, we have viewed geometric Langlands as a correspondence of sets

$$(5.3) \quad \{ {}^L G \text{ local systems } E \text{ on } X \} \leftrightarrow \{ \text{Hecke } D \text{ - modules } \text{Aut}_E \text{ on } \text{Bun}_G \}$$

We would like to promote 5.3 to a geometric/categorical level. Let  $\mathcal{L}_{OCLG}(X)$  denote the stack of  ${}^L G$ -local systems on  $X$ . The statement that we can attach to every  ${}^L G$ -oper a Hecke  $D$ -module on  $\text{Bun}_G$  is the statement that geometric Langlands holds for the locus of opers in  $\mathcal{L}_{OCLG}(X)$ . If we think of replacing a point  $\rho$  in  $\mathcal{L}_{OCLG}(X)$  by the corresponding skyscraper sheaf  $\mathcal{O}_\rho$ , we could imagine that there exists a transform between  $\mathcal{O}$ -modules on  $\mathcal{L}_{OCLG}(X)$  and  $D$ -modules on  $\text{Bun}_G$ , assigning to  $\rho$  is Hecke

$D$ -module  $M_\rho$ . In fact, viewing  $\mathcal{O}_\rho$  and  $M_\rho$  as objects in their corresponding derived categories, one arrives at the conjecture:

**Conjecture 1.** *There exists an isomorphism*

$$D^b(D_{\text{Bun}_G}) \cong D^b(\mathcal{O}_{\text{Loc}_G(X)})$$

In the case of  $GL_1$ , such an isomorphism was constructed by Laumon and Rothstein through a version of the Fourier-Mukai transform. One might hope that the general case may also be achieved through a geometric integral transform.

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